

Mathematics Assessment Project
CLASSROOM CHALLENGES
A Formative Assessment Lesson

Calculating Arcs and Areas of Sectors of Circles

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

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Calculating Arcs and Areas of Sectors of Circles

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to solve problems involving area and arc length of a sector of a circle using radians. It assumes familiarity with radians and should not be treated as an introduction to the topic. This lesson is intended to help you identify and assist students who have difficulties in:

- Computing perimeters, areas, and arc lengths of sectors using formulas.
- Finding the relationships between arc lengths, and areas of sectors after scaling.

COMMON CORE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

- A-CED: Create equations that describe numbers or relationships.
- A-SSE: Interpret the structure of expressions.
- A-REI: Solve equations and inequalities in one variable.
- G-C: Find arc lengths and areas of sectors of circles.

This lesson also relates to **all** of the *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 1, 2, 6, and 7:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

- Before the lesson, students work individually on an assessment task that is designed to reveal their current understanding and difficulties. You then review their solutions and create questions for students to consider, in order to improve their work.
- During the lesson, small groups work on a collaborative task matching cards according to the arc length, area, or perimeter of the sectors.
- After considering in the same small groups the general results of changing the radius and/or sector angle of a sector, there is a whole-class discussion.
- In a follow-up lesson, students review their initial solutions and then use what they have learned to either revise the same introductory assessment task or complete a different task.

MATERIALS REQUIRED

- Each student will need a copy of *Sectors of Circles* and *Sectors of Circles (revisited)*, the *Circles* sheet, a mini-whiteboard, pen, and eraser.
- Each small group of students will need a glue stick, an enlarged copy of *Arc Lengths and Areas of Sectors*, and the cut-up cards *Dominos 1*, *Dominos 2*, and *Changing the Angle and Radius of a Sector*.
- There is a projector resource to support whole-class discussions. The *Card Sets* may also be copied onto overhead transparencies to support these discussions.

TIME NEEDED

20 minutes before the lesson, a 100-minute lesson (or two shorter lessons), and 20 minutes in a follow-up lesson. Timings are approximate and will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: Sectors of Circles (20 minutes)

Have the students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the next lesson.

Give each student a copy of the *Sectors of Circles* task.

Read through the questions and try to answer them as carefully as you can.

It is important that students are allowed to answer the questions without your assistance, as far as possible.

Students should not worry too much if they cannot understand or do everything, because in the next lesson they will engage in a similar task, which should help them. Explain to students that by the end of the next lesson, they should expect to answer questions such as these confidently. This is their goal.

Assessing students' responses

Collect students' responses to the task and note down what their work reveals about their current levels of understanding and their different approaches.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores and will distract their attention from what they can do to improve their mathematics.

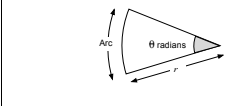
Instead, help students to make further progress by summarizing their difficulties as a list of questions. Some suggestions for these are given in the *Common issues* table on the next page. We suggest that you make a list of your own questions, based on your students' work, using the ideas on the following page. We recommend you:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these questions on the board when you return the work to the students.

Sectors of Circles

You may find the following formulas useful:



Arc length: $r\theta$

Area: $\frac{1}{2}r^2\theta$

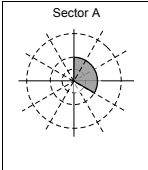
1. The diagram shows three concentric circles. The radii of the inner, middle, and outer circles are 2cm, 4cm and 8cm respectively. The circles are divided into twelve equal angles at the center. A sector of the middle circle is shaded.

a. Find the angle, in radians, of the shaded sector.

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Sector A

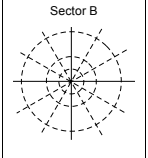
Sector B has the **same arc length** as Sector A, but has a different radius and sector angle.

b. Shade in Sector B. Explain how you know it has the **same arc length** as Sector A.

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Sector B

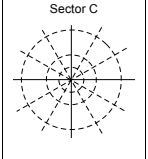
Sector C has the **same area** as Sector A, but a different radius and sector angle.

c. Shade in Sector C. Explain how you know it has the **same area** as Sector A.

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Sector C

2. The radius of Sector D is **half the radius** of Sector E. The area of Sector D is **half the area** of Sector E. Shade in possible sectors for D and E. Show all your work.

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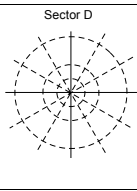
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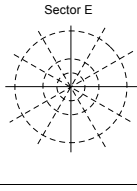
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Sector D



Sector E

Common issues:**Suggested questions and prompts:**

<p>Difficulty working in radians</p> <p>For example: The student figures out the size of the sector in degrees, but is unable to convert this angle to radians (Q1a).</p> <p>Or: The student incorrectly draws the angle.</p> <p>Or: The student calculates the arc length and area using degrees.</p>	<ul style="list-style-type: none"> • What is the definition of a radian? • How many radians are there at the center of a circle? • Now use the formula for arc length and area where the angle is measured in radians.
<p>Angles are greater than 2π</p> <p>For example: To obtain a sector with an area of $\frac{16\pi}{3}$ the student provides a radius of 2 cm and an angle of $\frac{8\pi}{3}$ radians (Q1c).</p>	<ul style="list-style-type: none"> • Does your answer make sense? • How many radians are there in one full turn of 360°?
<p>When substituting values into the formula for arc length or area of sector a technical error is made</p>	<ul style="list-style-type: none"> • Check your work.
<p>Answers are not given as multiples of π</p>	<ul style="list-style-type: none"> • Have you rounded your answers? • How can you write your answer more accurately?
<p>Assumes that because the sector radius of Sector D is half the radius of Sector E, the angles of each sector are equal (Q2)</p>	<ul style="list-style-type: none"> • Suppose a sector has a radius of 5 cm and another has a radius of 10 cm. Both have sector angles of π. What are their areas? Now figure out the areas for two different sectors. Make sure one sector radius is double the other one and both have the same sector angle. What do you notice? How can you use what you notice?
<p>Guess and check is used to figure out the size of the sector</p>	<ul style="list-style-type: none"> • Can you use a more efficient method? • How can the formulas help you solve this problem more efficiently?
<p>Little or no explanation is provided</p>	<ul style="list-style-type: none"> • Can you use a formula to explain your answer?

SUGGESTED LESSON OUTLINE

Whole-class interactive introduction (25 minutes)

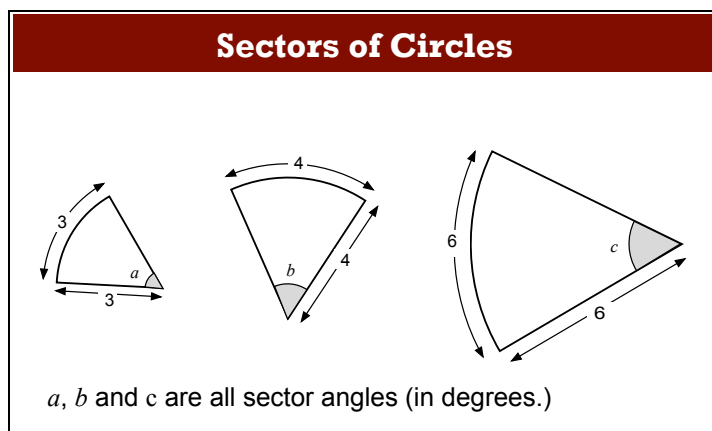
This lesson assumes that students are familiar with radians; it should not be regarded as an introduction to the topic.

Give each student a mini-whiteboard, pen, eraser, and a copy of the *Circles* handout. Students are to draw sectors on these circles.

Throughout the introduction, encourage students to first tackle a problem individually and only then discuss it with a neighbor. In this way students will have something to talk about. Maximize participation in the whole-class discussion by asking all students to show you solutions on their mini-whiteboards. Select a few students with interesting or contrasting answers to justify them to the class. Encourage the rest of the class to challenge these explanations.

Introducing the idea of a radian

Show Slide P-1 of the projector resource:



What do these sectors have in common?

Most students will reply that for all the sectors, the radius is equal to the arc length.

What else do they have in common? [Angles a , b , c are equal.]

How do you know this?

You may need to follow these questions up with:

What fraction of a complete circle is each sector? How do you know?

[Arc length of each sector = r , total circumference of circle = $2\pi r$,

so fraction of the complete circle is $\frac{r}{2\pi r} = \frac{1}{2\pi}$.]

What is the size of each angle? How do you know? [$360 \times \frac{1}{2\pi} = \frac{360}{2\pi} = 57.3^\circ$]

Students may notice that the sectors are all similar.

Each of these angles is equivalent to one radian.

Today you are going to work with sectors of circles, using radians rather than degrees to describe the angles.

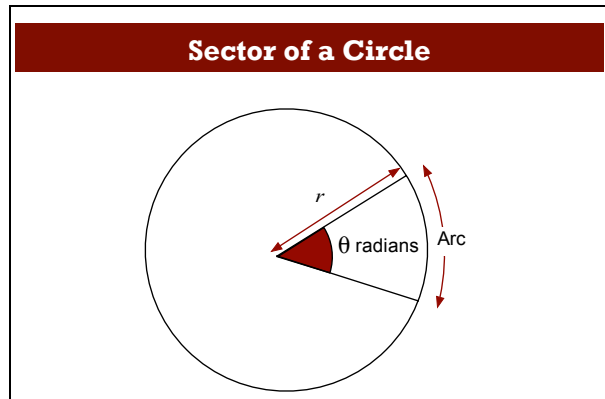
Spent a few minutes checking that students understand the relationship between radius, arc length, and sector angle:

*On your mini-whiteboard, show me another sector with an angle of one radian.
Write down the radius and arc length of your sector.
Now show me a sector with an angle of 2 radians, 3 radians, 1.5 radians ...*

Introduce the formula for arc length

First, verify that all students remember how to calculate the circumference and area of a full circle.

Show Slide P-2 of the projector resource:



Then ask:

How many radians are there in a full turn of 360°? [$2\pi = \text{circumference} \div r$.]

On the Circles handout, shade in a sector with an angle of π radians.

Now repeat for 2π , $\pi/2$, $\pi/6$, $5\pi/6$... radians.

Depending on your class, you may want to ask students to shade in more than these five examples of sector angles.

What did you make the radii of your circles? Did this matter?

[No. The radii can be any length, as they are unrelated to the angles.]

What fraction of the circumference of the full circle is each arc length? How do you know?

Suppose your circle has a radius of r . What is the arc length of each sector?

Project Slides P-3, P-4, and P-5 in turn, which show the table below (partially filled) and complete the cells with the class:

Sector angle	π	2π	$\frac{\pi}{2}$	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	θ
Fraction of circle	$\frac{1}{2}$	1	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{\theta}{2\pi}$
Arc length	πr	$2\pi r$	$\frac{\pi r}{2}$	$\frac{\pi r}{6}$	$\frac{5\pi r}{6}$?

Challenge students to generalize the results in the last column.

What is the formula for figuring out the arc length of this sector?

How do you know?

What do you multiply the angle by to obtain the arc length?

These last few questions are challenging so allow students sufficient time to think about it on their own and then discuss it with a partner. Encourage students to use the arc lengths they have already calculated.

The arc length is $\frac{\theta}{2\pi}$ of the circumference of the circle, so the arc length = $\frac{\theta}{2\pi} \times 2\pi r = r\theta$

Introducing the formula for area

Now move your questioning on to deriving a formula for the area of a sector.

Project the Slides P-6, P-7, and P-8 as the following discussion unfolds:

What is the area of your sectors?

Sector angle	π	2π	$\frac{\pi}{2}$	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	θ
Fraction of circle	$\frac{1}{2}$	1	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{\theta}{2\pi}$
Area of sector	$\frac{1}{2}\pi r^2$	πr^2	$\frac{\pi r^2}{4}$	$\frac{\pi r^2}{12}$	$\frac{5\pi r^2}{12}$?

How can we figure out the general formula for a sector of a circle of radius r and sector angle θ ?

What expression do you multiply the angle by to obtain the sector area?

Again this is a challenging problem so allow students sufficient time.

$$\text{Area of sector} = \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2}\theta r^2$$

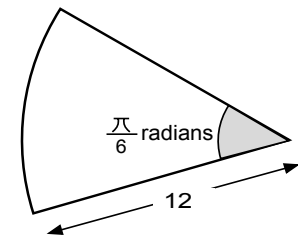
Check that students understand through questioning

Now draw on the board a sector with radius 12 and sector angle of $\frac{\pi}{6}$.

To establish that students understand the difference between perimeter and arc length, you may want to ask:

Is the perimeter always bigger than the arc length? Why?

[Yes, because you add the radius twice.] What is the perimeter of this sector? $[24+2\pi.]$



Ask the following questions in turn. Students are to draw their sectors on their mini-whiteboards.

Show me a sector that has the same arc length as this one, but with a different radius and sector angle.

Show me a sector with double the perimeter. How do you know?

Show me a sector with double the area. How do you know?

If students provide decimal answers (e.g. 6.283185), you could ask:

Which is the more accurate answer 2π or 6.283185?

In this lesson some measurements will be provided as multiples of π . Leave your answers as multiples of π .

Collaborative activity 1: Domino Cards (30 minutes)

Organize the class into groups of two or three students. Give each group the cards *Dominos 1* and *Dominos 2*.

Explain to the class that they are about to link cards together so that they will eventually create a closed loop.

If students are unsure what to do, explain as follows:

In front of you are six dominos.

Drawn on each domino is a diagram with three concentric circles.

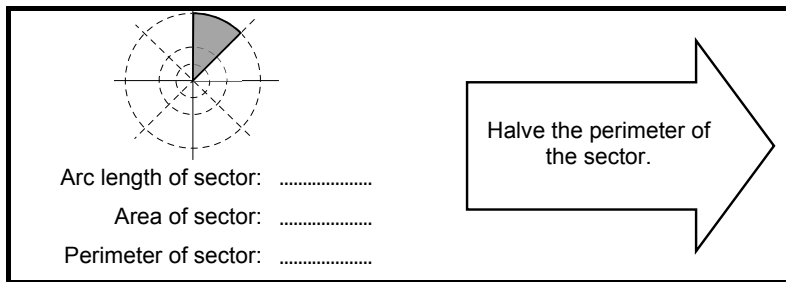
The circles are divided into eight equal angles at the center.

The radius of the biggest circle is double the radius of the middle one.

The radius of the middle circle is double the radius of the smallest one.

*We **don't** know how big these radii are. You will need to work them out.*

Choose a domino.



You must find the missing lengths and areas and draw any missing sectors.

Can you work out any of the missing answers? If not choose another domino!

Next to the diagram is an arrow with an instruction.

You must connect the dominos by following the instructions.

Show Slide P-9 of the projector resource and ask students to read it carefully:

Matching the Dominos

Moving out from the center, the radius of each circle is doubled. The radii are all integer lengths.
All the center angles are equal.

Apply these instructions to the shaded sector on the domino. Then match the resultant sector to a sector on another domino.

Complete this information as you work through the task.
Leave your answers as multiples of π .

Make sure every person in your group understands and can explain the placement of each domino.

Work together to connect all the dominos.

As you do this, discuss with your partner how you came to your decision.

Both of you need to agree on and explain all the connections.

Start by finding cards that give you plenty of information about the circles and the sectors.

The purpose of this structured group work is to help students engage with each other's explanations and take responsibility for each other's understanding.

If students assume the radii are the same size as the radii in the assessment (2 cm, 4 cm, and 8 cm) or if students struggle to get started on the task, prompt them by asking:

*Before connecting the dominos, what do you first need to know? [The radii of each circle.]
Are there any cards that will give you this information?*

If students are still struggling direct them to **Cards D or F**.

*What do you know about this sector?
How can you use this information to help you to find the radius?*

During small group work, you have two tasks, to notice students' approaches to the task and to support student reasoning.

Note different student approaches to the task

Notice how students make a start on the task, where they get stuck, and how they respond if they do come to a halt. How do they approach the task? Do they first try to figure out the unknown measures on each card? Do they try to work in degrees? Are they able to substitute values into formula correctly? You can use this information to focus a whole-class discussion towards the end of the lesson.

Support student reasoning

Try not to make suggestions that move students towards a particular approach to the task. Instead, ask questions that help students clarify their thinking. Draw on the questions in the *Common issues* table to support your own questioning.

If the whole-class is struggling on the same issue, write relevant questions on the board and hold an interim discussion.

Check that each member of the group understands and can explain each card placement. If you find a student in any group is struggling to respond to your questions, return in a few minutes, to check how the group has together worked on their shared understanding.

*Which card(s) provides you with information about the radius of the sector? [Cards D, F, and E.]
What do you know about this sector? How can you use this to figure out the radius?
What does the perimeter tell you about the radius/arc length?*

*Which cards provide you with the least information? How will you deal with these cards?
[Leave them to the end.]*

If the perimeter is halved/doubled, what happens to the length of the radius? What happens to the arc length?

If the arc length remains the same, does this mean the radius and the sector angle remains the same?

If the radius increases, how will the sector angle change?

Sharing work (10 minutes)

As students finish connecting the cards, ask them to share their solution with a neighboring group.

Slide P-10 of the projector resource summarizes how students should work:

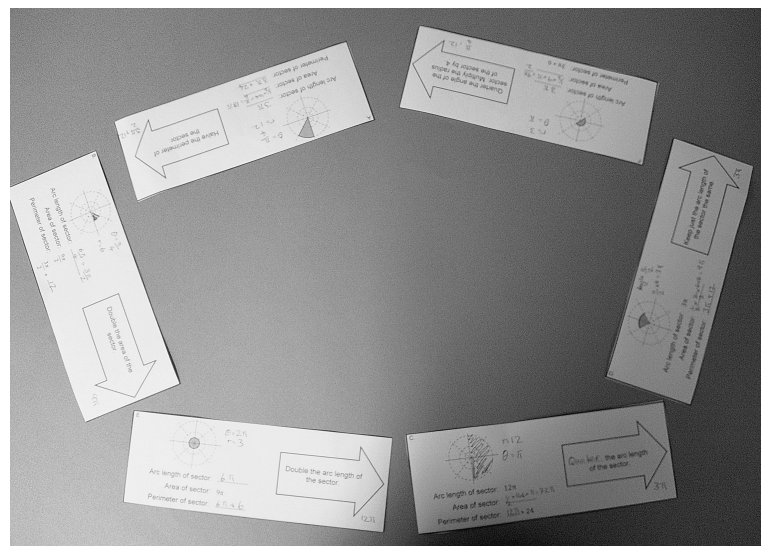
Sharing work

1. Check to see which matches are different from your own.
2. A member of each group needs to explain their reasoning for these matches. If anything is unclear, ask for clarification.
3. Then together consider if you should change any of your answers.

It is important that everyone in both groups understands the math.

You are responsible for each other's learning.

When students are satisfied with their loop of dominos, give each small group a glue stick. They are to glue the cards onto the poster. Their poster may look something like this:



Collaborative activity 2: Changing Arc Lengths and Areas (20 minutes)

This activity provides students with an opportunity to summarize and generalize their work in the previous task.

Give each group the sheet *Arc Lengths and Areas of Sectors*, and the cards *Changing the Angle and Radius of a Sector*.

You are to place two of the small instruction cards into one of the boxes on your sheet.

Students may find their domino connections help them with this task. Support the students as before.

When students are satisfied with their placements, they glue the cards onto the sheet.

Encourage students who work through the task quickly to think about how they can explain the scaling in general terms. They may use algebra in their explanation, or simply highlight the properties of a formula that determine the scaling.

Can you use algebra to show you are correct?

If the radius is multiplied by n , what should happen to θ in order to keep the arc length the same/to double the arc length?

If the radius is multiplied by n , what should happen to θ in order to keep the area the same/to double the area?

Whole-class discussion (15 minutes)

Depending on how your class has progressed you may want to focus the discussion on the first or second collaborative activity.

Organize a whole-class discussion about different strategies used to connect (or place) the cards. First select a pair of cards that most groups connected (or placed) correctly. This approach may encourage good explanations. Then select one or two cards that most groups found difficult to connect (or place).

Once one group has justified their choice for a particular connection (or placement), ask other students to contribute ideas of alternative approaches and their views on which reasoning method was easier to follow. The intention is that you focus on getting students to understand and share their **reasoning**, not just checking that everyone produced the right answers. Use your knowledge of the students' individual and group work to call on a wide range of students for contributions. You may want to draw on the questions in the *Common issues* table to support your own questioning and use a selection of these questions:

*Johnny, do you agree with Maddie's connection (or placement) of the cards?
Can you put her explanation into your own words?*

*Which was the easiest card to connect (or place)? Why?
Does anyone disagree with the connection (or placement)?*

*Which was the most difficult card to connect (or place)? Why?
Does anyone disagree with the connection (or placement)?*

When sharing your work with another group, did any of you see anything to make you change your mind? Why?

You may also want to compare two sector areas and arc lengths:

*Can anyone think of another way of keeping the sector arc length/area the same?
How many ways are there of keeping the sector arc length/area the same?*

*Can anyone think of another way of doubling the sector arc length/area?
How many ways are there of doubling the sector arc length/area?*

Follow-up lesson: reviewing the assessment task (20 minutes)

Return to the students their original assessment *Sectors of Circles*.

If you have not added questions to individual pieces of work, then write your list of questions on the board. Students should select from this list only the questions they think are appropriate to their own work.

Read through your original solutions to the task and think about what you have learned this lesson.

Carefully read through the questions I have written.

Spend a few minutes trying to answer these questions.

Students can either revise the pre-assessment task or attempt the new task. If students struggled with the original assessment then they may benefit from revising this assessment. In order that students can see their own progress, ask them to complete the task using a different color pen or give them a

second blank copy of the task. Otherwise, give students a copy of the *Sector of Circles (revisited)* task.

Use what you have learned to complete the new assessment task/revise your answers.

You could give this as a homework task.

SOLUTIONS

Assessment task: Sectors of Circles

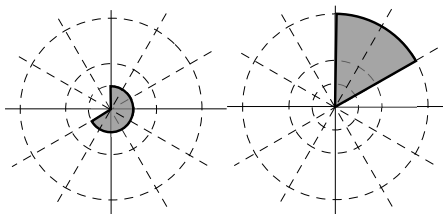
1. a. The sector angle is $\frac{2\pi}{3}$.

b. Sector A:

$$\text{Arc length} = 4 \times \frac{2\pi}{3} = \frac{8\pi}{3}.$$

Sector B:

$$r = 2; \quad \theta = \frac{4\pi}{3} \quad \text{OR} \quad r = 8; \quad \theta = \frac{\pi}{3}.$$

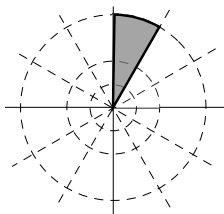


c. Sector A:

$$\text{Area} = \frac{1}{2} \times 4^2 \times \frac{2\pi}{3} = \frac{16\pi}{3}.$$

Sector C:

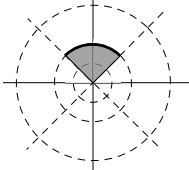
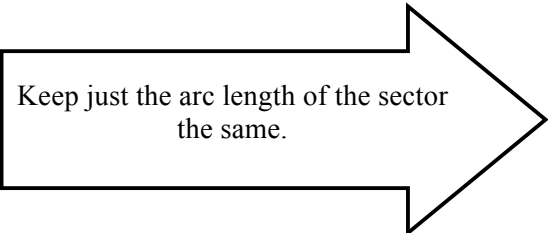
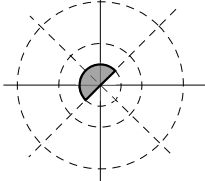
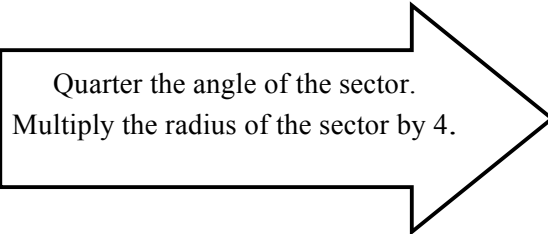
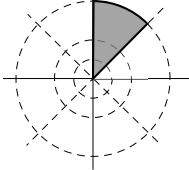
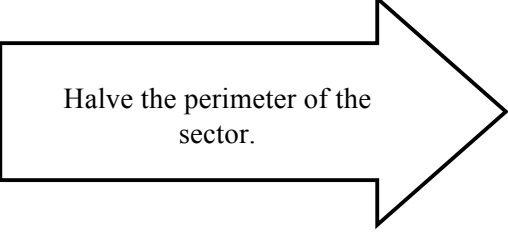
$$r = 8; \quad \theta = \frac{\pi}{6}.$$



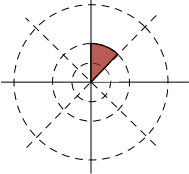
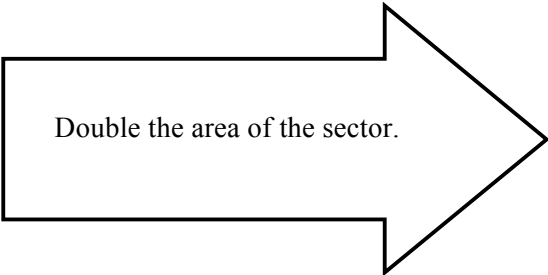
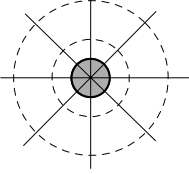
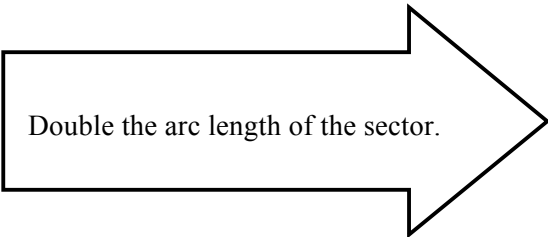
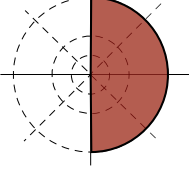
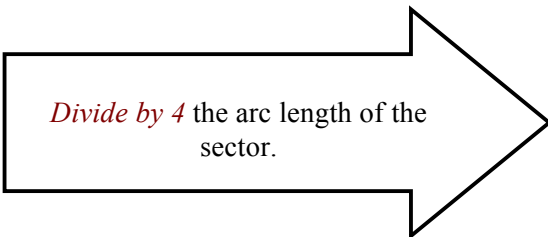
2. The sector angle of Sector E should be half the sector angle of Sector D.

Collaborative activity 1: Domino Cards

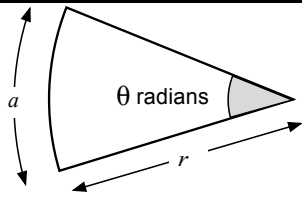
Radii of circles are 3 cm, 6 cm, and 12 cm.

<p>D.</p>  <p>Arc length of sector: $3\pi \Rightarrow \text{radius} = 6; \theta = \frac{\pi}{2}$</p> <p>Area of sector: $\frac{1}{2} \times \frac{\pi}{2} \times 6^2 = 9\pi$</p> <p>Perimeter of sector: $3\pi + 12$</p>	 <p>Keep just the arc length of the sector the same.</p>
<p>F.</p>  <p>Perimeter of sector: $3\pi + 6 \Rightarrow \text{radius} = 3$</p> <p>Arc length of sector: $3\pi \Rightarrow \theta = \pi$</p> <p>Area of sector: $\frac{1}{2} \times \pi \times 3^2 = \frac{9\pi}{2}$</p>	 <p>Quarter the angle of the sector. Multiply the radius of the sector by 4.</p>
<p>A.</p>  <p>radius = 12; $\theta = \frac{\pi}{4}$</p> <p>Arc length of sector: $12 \times \frac{\pi}{4} = 3\pi$</p> <p>Perimeter of sector: $3\pi + 24$</p> <p>Area of sector: $\frac{1}{2} \times \frac{\pi}{4} \times 12^2 = 18\pi$</p>	 <p>Halve the perimeter of the sector.</p>

Collaborative activity 1: Domino Cards (continued)

<p>B.</p>  <p>Arc length of sector: $\frac{3\pi}{2}$</p> <p>Area of sector: $\frac{9\pi}{2} \Rightarrow$ Radius : 6; $\theta = \frac{\pi}{4}$</p> <p>Perimeter of sector: $\frac{3\pi}{2} + 12$</p> <p>Students may use simultaneous equations to figure out the missing measures :</p> $\frac{1}{2}\theta r^2 = \frac{9\pi}{2}$ $\theta r = \frac{3\pi}{2} \Rightarrow \theta = \frac{3\pi}{2r}$ $\frac{1}{2} \times \frac{3\pi}{2r} \times r^2 = \frac{9\pi}{2}$ $r = 6$ $\theta = \frac{3\pi}{2 \times 6} = \frac{\pi}{4}$	 <p>Double the area of the sector.</p>
<p>E.</p>  <p>Area of sector : 9π</p> <p>Radius:3; $\theta = 2\pi$</p> <p>Arc length of sector:6π</p> <p>Perimeter of sector: $6\pi + 6$</p>	 <p>Double the arc length of the sector.</p>
<p>C.</p>  <p>Arc length of sector: 12π</p> <p>Perimeter of sector: $12\pi + 24 \Rightarrow$ Radius:12; $\theta = \pi$</p> <p>Area of sector : $\frac{1}{2} \times \pi \times 12^2 = 72\pi$</p>	 <p><i>Divide by 4</i> the arc length of the sector.</p>

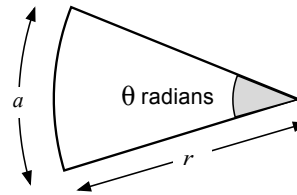
Collaborative activity 2: Changing Arc Lengths and Areas



Another sector with the same arc length, can be drawn by:

doubling the angle θ , and halving the radius, r .

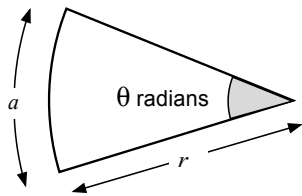
multiplying the angle θ by 4, and dividing the radius, r by 4.



Another sector with the same area, can be drawn by:

multiplying the angle θ by 4, and halving the radius, r .

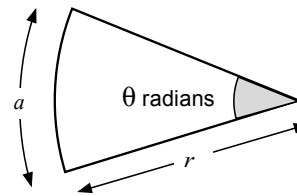
dividing the angle θ by 4, and doubling the radius, r .



Another sector with double the arc length, can be drawn by:

keeping the angle, θ , the same, and doubling the radius, r .

doubling the angle θ , and keeping the radius the same.



Another sector with double the area, can be drawn by:

halving the angle θ , and doubling the radius, r .

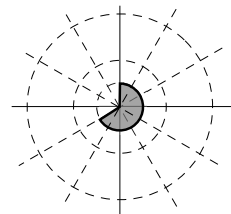
doubling the angle θ , and keeping the radius the same.

Assessment task: Sectors of Circles (revisited)

1. a. $\theta = \frac{4\pi}{3}$

$$\text{arc length} = \frac{4\pi}{3} = r \times \frac{4\pi}{3}$$

$$r = 1$$



b. The radius can be doubled :

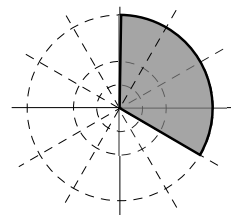
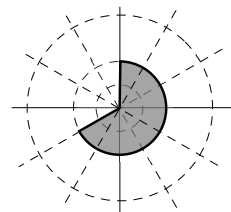
$$r = 2; \quad \theta = \frac{4\pi}{3};$$

$$\text{Arc length} = 2 \times \frac{4\pi}{3} = \frac{8\pi}{3}.$$

Or the radius can be increased by a factor of 4 and the size of the angle halved :

$$r = 4; \quad \theta = \frac{2\pi}{3};$$

$$\text{Arc length} = 4 \times \frac{2\pi}{3} = \frac{8\pi}{3}.$$



c. Area of Sector A = $\frac{1}{2} \times 1^2 \times \frac{4\pi}{3} = \frac{2\pi}{3}$.

$$\text{Area of Sector C} = \frac{4\pi}{3}.$$

The radius can be doubled and the size of the angle halved :

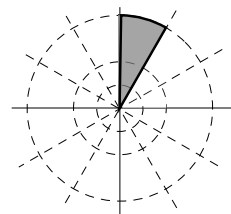
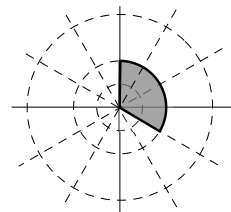
$$r = 2; \quad \theta = \frac{2\pi}{3};$$

$$\text{Area} = \frac{1}{2} \times 2^2 \times \frac{2\pi}{3} = \frac{4\pi}{3}.$$

Or the radius can be increased by a factor of 4 and the size of the angle decreased by a scale factor of 8 :

$$r = 4; \quad \theta = \frac{\pi}{6};$$

$$\text{Area} = \frac{1}{2} \times 4^2 \times \frac{\pi}{6} = \frac{4\pi}{3}.$$



2. In general terms, for two sectors to have equal areas but differing radii and sector angles then if the radius of the first sector is r and its sector angle is θ , then the radius of the second sector could be nr and its sector angle θ/n^2 n is any positive number.

In this case, the two sectors could have dimensions :

$$r = 1; \quad \theta = \frac{2\pi}{3} \quad \text{and} \quad r = 2; \quad \theta = \frac{\pi}{6}.$$

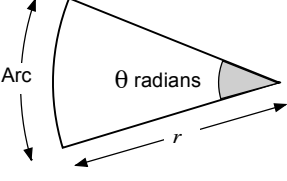
OR

$$r = 2; \quad \theta = 2\pi; \quad \text{and} \quad r = 4; \quad \theta = \frac{\pi}{2}.$$

And so on.

Sectors of Circles

You may find the following formulas useful:



Arc length: $r\theta$

Area: $\frac{1}{2}r^2\theta$

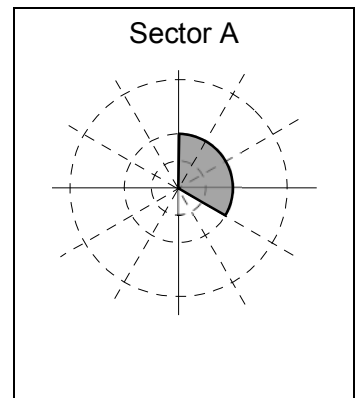
1. The diagram shows three concentric circles. The radii of the inner, middle, and outer circles are 2cm, 4cm and 8cm respectively. The circles are divided into twelve equal angles at the center. A sector of the middle circle is shaded.

- a. Find the angle, in radians, of the shaded sector.

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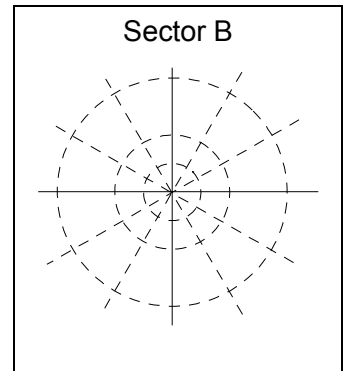
Sector B has the **same arc length** as Sector A, but has a different radius and sector angle.

- b. Shade in Sector B.
Explain how you know it has the **same arc length** as Sector A.

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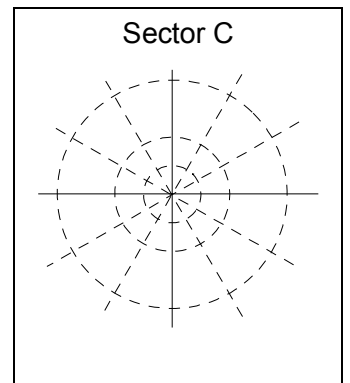
Sector C has the **same area** as Sector A, but a different radius and sector angle.

- c. Shade in Sector C.
Explain how you know it has the **same area** as Sector A.

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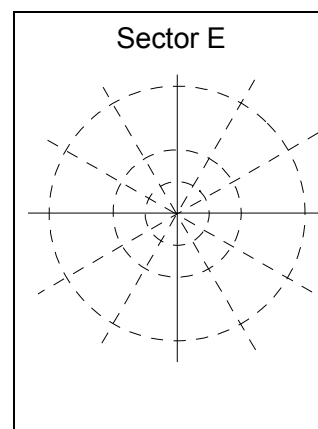
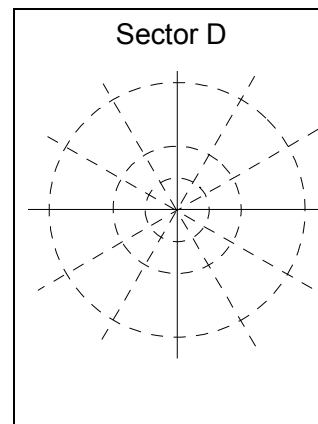


2. The radius of Sector D is **half the radius** of Sector E.

The area of Sector D is **half the area** of Sector E.

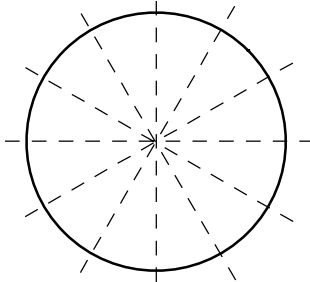
Shade in possible sectors for D and E.

Show all your work.

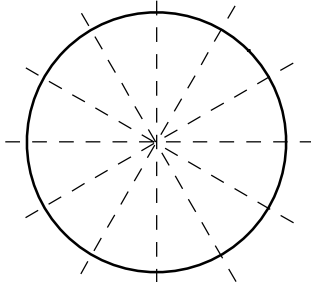


Circles

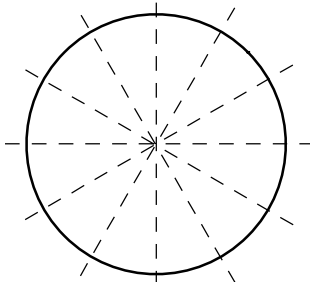
Sector Angle =



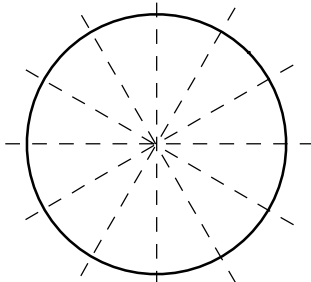
Sector Angle =



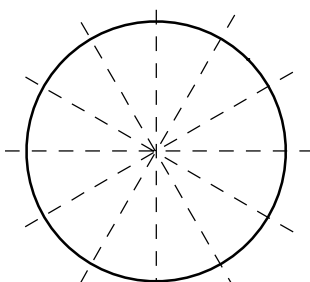
Sector Angle =



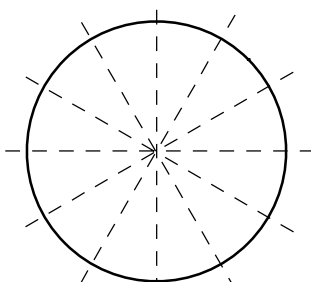
Sector Angle =



Sector Angle =

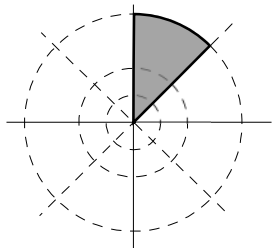


Sector Angle =



Card Set: Dominos 1

A.



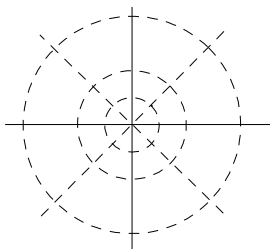
Arc length of sector:

Area of sector:

Perimeter of sector:

Halve the perimeter of
the sector.

B.



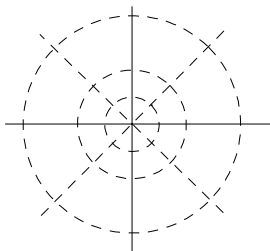
Arc length of sector:

Area of sector: $\frac{9\pi}{2}$

Perimeter of sector: $\frac{3\pi}{2} + \dots$

Double the area of the
sector.

C.



Arc length of sector: 12π

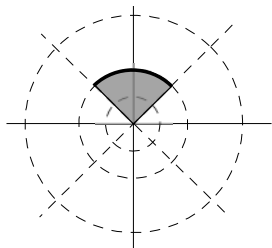
Area of sector:

Perimeter of sector: + 24

..... the arc length
of the sector.

Card Set: Dominos 2

D.



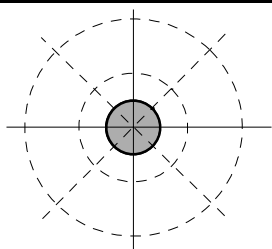
Arc length of sector: 3π

Area of sector:

Perimeter of sector:

Keep just the arc length of
the sector the same.

E.



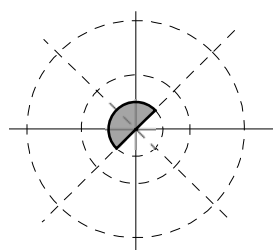
Arc length of sector:

Area of sector: 9π

Perimeter of sector:

Double the arc length of
the sector.

F.



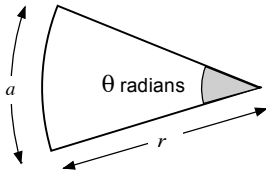
Arc length of sector:

Area of sector:

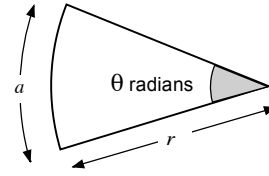
Perimeter of sector: $3\pi + 6$

Quarter the angle of the
sector. Multiply the radius
of the sector by 4.

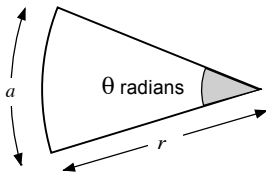
Arc Lengths and Areas of Sectors



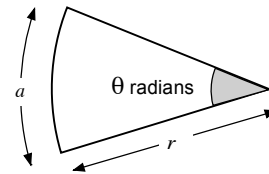
Another sector with the same arc length, can be drawn by:



Another sector with the same area, can be drawn by:



Another sector with double the arc length, can be drawn by:



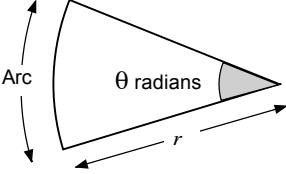
Another sector with double the area, can be drawn by:

Card Set: Changing the Angle and Radius of a Sector

multiplying the angle θ by 4, and halving the radius, r .	doubling the angle θ , and keeping the radius the same.
multiplying the angle θ by 4, and dividing the radius, r by 4.	keeping the angle, θ , the same, and doubling the radius, r .
doubling the angle θ , and halving the radius, r .	dividing the angle θ by 4, and doubling the radius, r .
doubling the angle θ , and keeping the radius the same.	halving the angle θ , and doubling the radius, r .

Sectors of Circles (revisited)

You may find the following formulas useful.

	<p style="text-align: right;">Arc length: $r\theta$</p> <p style="text-align: right;">Area: $\frac{1}{2}r^2\theta$</p>
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1. The diagram shows three concentric circles. The radii of the inner, middle, and outer circles are 1cm, 2cm and 4cm respectively. The circles are divided into twelve equal angles at the center.

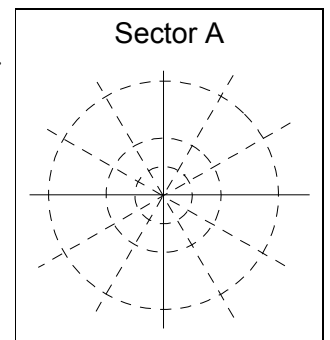
- a. Draw Sector A. It has a sector angle of $\frac{4\pi}{3}$ and an arc length of $\frac{4\pi}{3}$.

Explain your sector.

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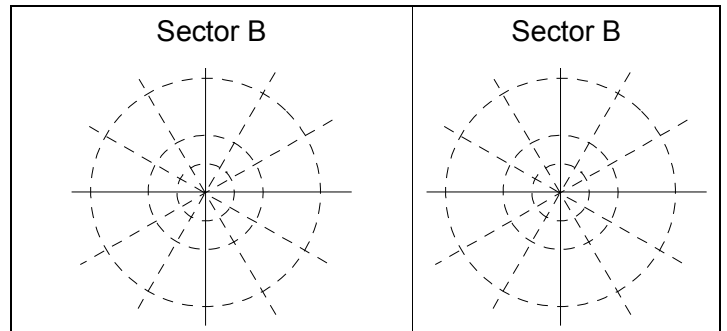
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- b. Sector B has **double** arc length of Sector A. The radius of Sector B is **different** from the radius of Sector A. What does Sector B look like?

Draw two different answers to this question.

Show your reasoning.



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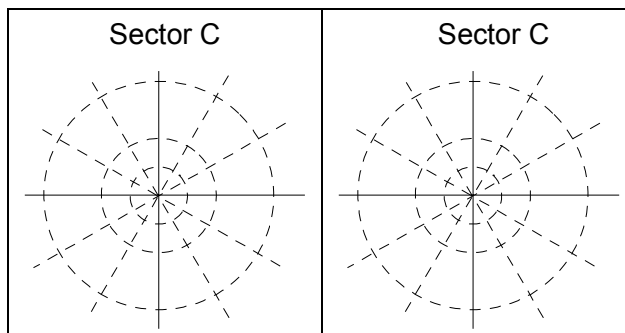
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- c. Sector C has **double** the area of Sector A.
 The radius of Sector C is **different** from the radius of Sector A.
 What does Sector C look like?

Draw two different answers to this question.

Show your reasoning.



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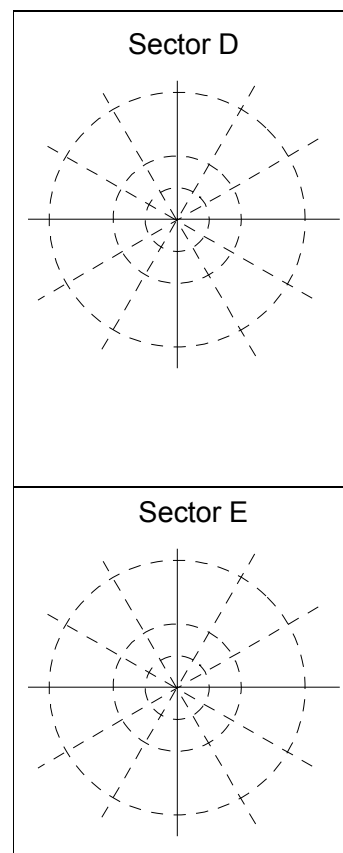
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2. The area of Sector D is **the same as** the area of Sector E.
 The angle and radius of Sector D are both **different** from the angle and radius of Sector E.

Shade in possible sectors for D and E.

Show your reasoning.



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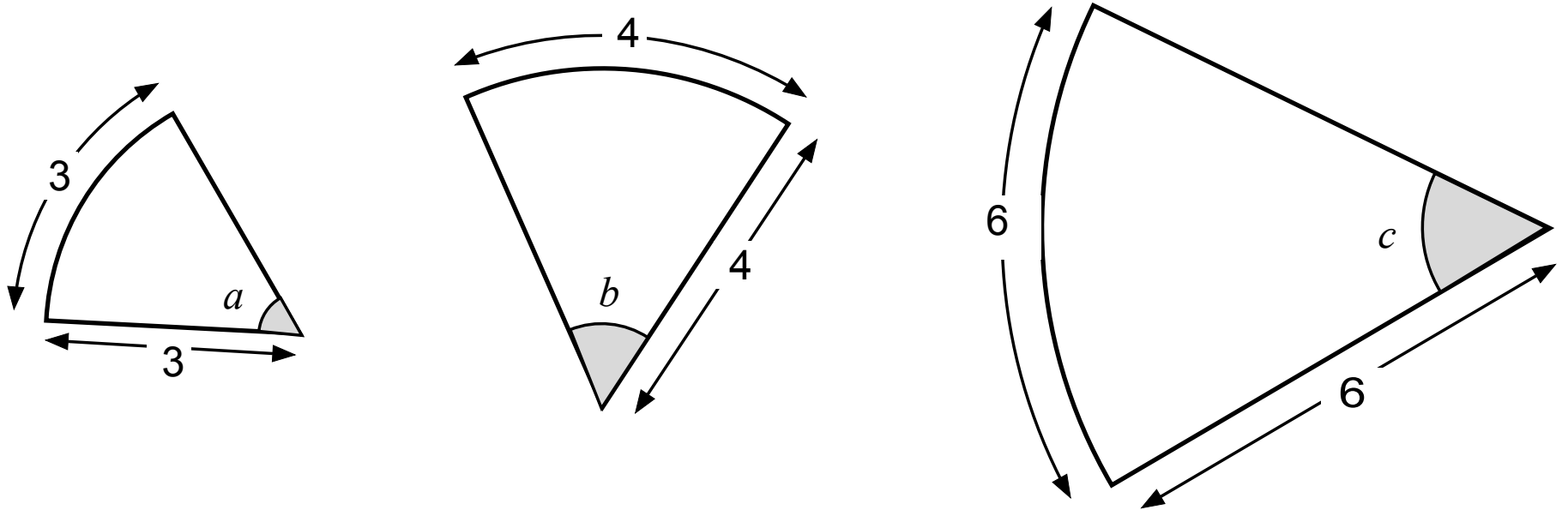
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Calculating Arcs and Areas of Sectors of Circles

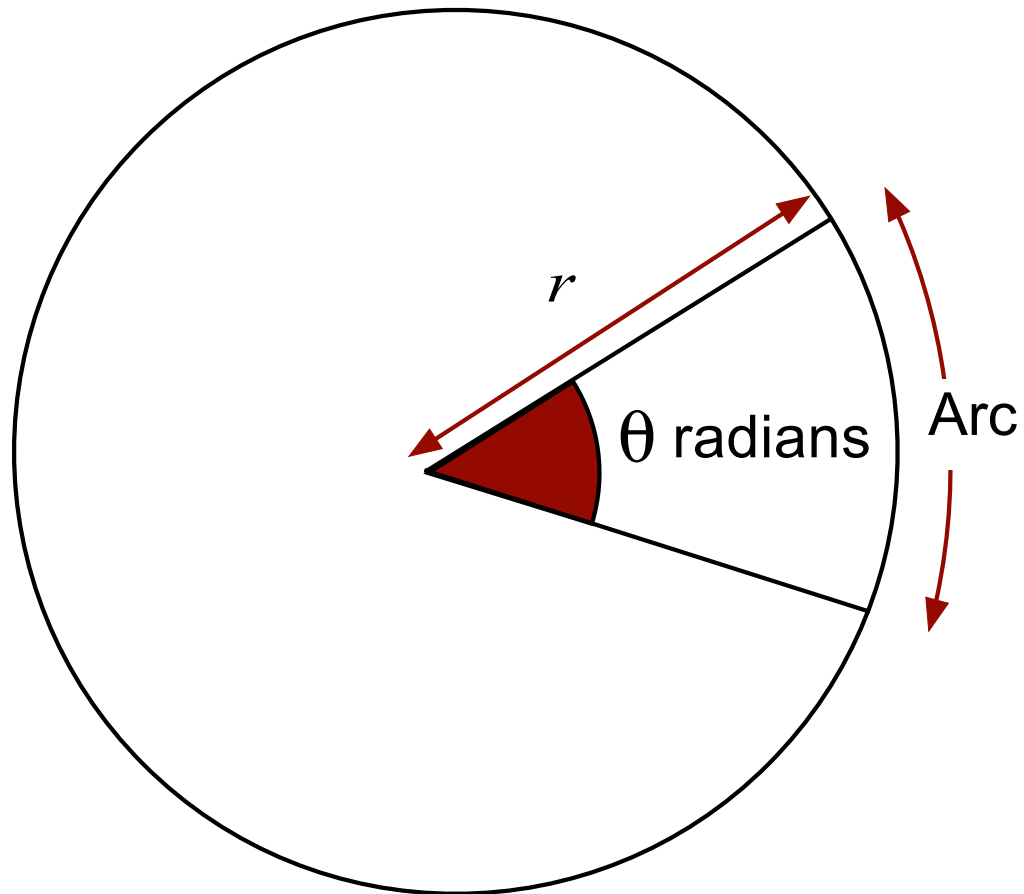
Projector Resources

Sectors of Circles



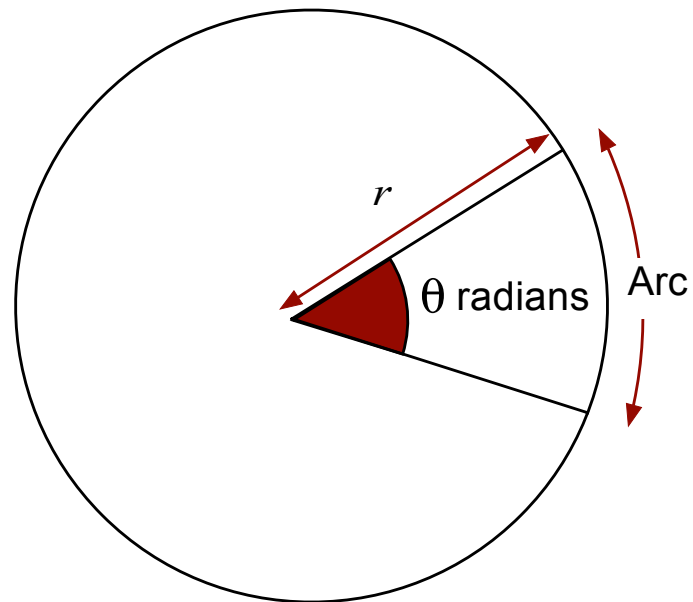
a , b and c are all sector angles (in degrees.)

Sector of a Circle



What fraction of a circle?

Sector angle	π	2π	$\frac{\pi}{2}$	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	θ
Fraction of circle						



What arc length?

Sector angle	π	2π	$\frac{\pi}{2}$	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	θ
Fraction of circle	$\frac{1}{2}$	1	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{\theta}{2\pi}$
Arc length						

What area of sector?

Sector angle	π	2π	$\frac{\pi}{2}$	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	θ
Fraction of circle	$\frac{1}{2}$	1	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{\theta}{2\pi}$
Arc length	πr	$2\pi r$	$\frac{\pi r}{2}$	$\frac{\pi r}{6}$	$\frac{5\pi r}{6}$?

What area of sector?

Sector angle	π	2π	$\frac{\pi}{2}$	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	θ
Fraction of circle	$\frac{1}{2}$	1	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{\theta}{2\pi}$
Arc length	πr	$2\pi r$	$\frac{\pi r}{2}$	$\frac{\pi r}{6}$	$\frac{5\pi r}{6}$	$r\theta$
Area of sector						

Generalize?

Sector angle	π	2π	$\frac{\pi}{2}$	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	θ
Fraction of circle	$\frac{1}{2}$	1	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{\theta}{2\pi}$
Arc length	πr	$2\pi r$	$\frac{\pi r}{2}$	$\frac{\pi r}{6}$	$\frac{5\pi r}{6}$	$r\theta$
Area of sector	$\frac{1}{2}\pi r^2$	πr^2	$\frac{\pi r^2}{4}$	$\frac{\pi r^2}{12}$	$\frac{5\pi r^2}{12}$?

Formulas for general case

Sector angle	π	2π	$\frac{\pi}{2}$	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	θ
Fraction of circle	$\frac{1}{2}$	1	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{\theta}{2\pi}$
Arc length	πr	$2\pi r$	$\frac{\pi r}{2}$	$\frac{\pi r}{6}$	$\frac{5\pi r}{6}$	$r\theta$
Area of sector	$\frac{1}{2}\pi r^2$	πr^2	$\frac{\pi r^2}{4}$	$\frac{\pi r^2}{12}$	$\frac{5\pi r^2}{12}$	$\frac{1}{2}r^2\theta$

Matching the Dominos

Moving out from the center, the radius of each circle is doubled. The radii are all integer lengths. All the center angles are equal.

Apply these instructions to the shaded sector on the domino. Then match the resultant sector to a sector on another domino.



Complete this information as you work through the task. Leave your answers as multiples of π .

Make sure every person in your group understands and can explain the placement of each domino.

Sharing work

1. Check to see which matches are different from your own.
2. A member of each group needs to explain their reasoning for these matches. If anything is unclear, ask for clarification.
3. Then together consider if you should change any of your answers.

It is important that everyone in both groups understands the math.

You are responsible for each other's learning.

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
Shell Center Team at the Center for Research in Mathematical Education
University of Nottingham, England:

Malcolm Swan,
Nichola Clarke, Clare Dawson, Sheila Evans, Colin Foster, and Marie Joubert
with
Hugh Burkhardt, Rita Crust, Andy Noyes, and Daniel Pead

We are grateful to the many teachers and students, in the UK and the US,
who took part in the classroom trials that played a critical role in developing these materials

The classroom observation teams in the US were led by
David Foster, Mary Bouck, and Diane Schaefer

This project was conceived and directed for
The Mathematics Assessment Resource Service (MARS) by
Alan Schoenfeld at the University of California, Berkeley, and
Hugh Burkhardt, Daniel Pead, and Malcolm Swan at the University of Nottingham

Thanks also to Mat Crosier, Anne Floyde, Michael Galan, Judith Mills, Nick Orchard, and Alvaro Villanueva who contributed to the design and production of these materials

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We are particularly grateful to
Carina Wong, Melissa Chabran, and Jamie McKee

The full collection of Mathematics Assessment Project materials is available from

<http://map.mathshell.org>