CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Sorting Equations of Circles 2

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Sorting Equations of Circles 2

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Translate between the equations of circles and their geometric features.
- Sketch a circle from its equation.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

G-GPE: Translate between the geometric description and the equation for a conic section.
Use coordinates to prove simple geometric theorems algebraically
F-IF: Interpret functions that arise in applications in terms of the context.
A-CED: Create equations that describe numbers or relationships.

This lesson also relates to **all** of the *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 1, 2, 6, and 7:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

- Before the lesson, students work individually on an assessment task that is designed to reveal their current understanding and difficulties. You then review their work and create questions for students to answer in order to improve their solutions.
- During the lesson, students work in small groups on a collaborative task, categorizing equations and geometric descriptions of circles.
- At the end of the lesson there is a whole-class discussion.
- In a follow-up lesson, students review their initial solutions to the assessment task and then use what they have learned to either revise the same task or complete a different task.

MATERIALS REQUIRED

- Each student will need copies of the assessment tasks *Going Round in Circles 2* and *Going Round in Circles 2 (revisited)*, a mini-whiteboard, pen, and eraser.
- Each small group of students will need the cut up *Table Headings: Categorizing Equations*, *Card Set: Equations 1* and *Card Set: Equations 2,* a large sheet of paper for making a poster, and a glue stick. You may want to copy the card sets onto transparencies to be used on an overhead projector to support whole-class discussions.
- There is also a projector resource to support whole-class discussions.

TIME NEEDED

20 minutes before the lesson, a 90-minute lesson (or two shorter lessons), and 20 minutes in a follow-up lesson. Timings are approximate and will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *Going Round in Circles 2* (20 minutes)

Have students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the next lesson.

Give each student a copy of the assessment task *Going Round in Circles 2*.

> *Read through the questions and try to answer them as carefully as you can.*

It is important that, as far as possible, students are allowed to answer the questions without your assistance.

Students should not worry too much if they cannot understand or do everything because, in the next lesson, they will engage in a similar task, which should help them. Explain to students that by the end of the next lesson, they should expect to be able to answer questions such as these confidently. This is their goal.

Going Round in Circles 2

1. Complete the drawing of the graph of the equation $(x + 1)^2 + (y - 3)^2 = 9$ to show the x- axis and y-axis. Add numbers to each of these axes.

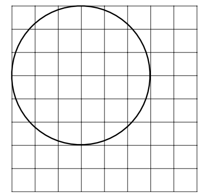


Figure out the co-ordinates of any x-intercepts and y-intercepts.

Explain your answer(s).

2. Write an equation of a circle that has two x-intercepts but just one y-intercept and a radius of 6.

Explain your answer.

Assessing students' responses

Collect students' responses to the task and note down what their work reveals about their current levels of understanding and their different approaches.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a list of questions. Some suggestions for these are given in the *Common issues* table on the next page. We suggest that you make a list of your own questions, based on your students' work, using the ideas on the following page. We recommend you:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these questions on the board when you return the work to the students in the follow-up lesson.

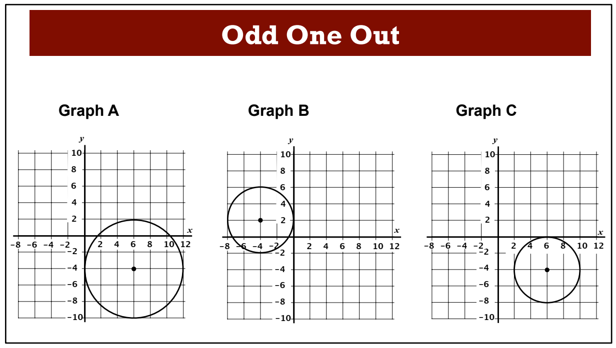
Common issues:	**Suggested questions and prompts:**
Cannot get started (Q1)	• What does the equation tell you about the features of its graph? • Try sketching a graph on your own axes.
Adds the coordinates of the center of the circle but the axes are not drawn (Q1)	• Now draw in axes to fit your center.
Assumes that the origin is the center of the circle (Q1)	• How can you check your center is correct?
Incorrect identification of the position of the axes (Q1)	• Substitute $y = 0$ into the equation. Does this agree with your drawing? • Substitute $x = 0$ into the equation. Does this agree with your drawing?
Pythagorean theorem incorrectly used when figuring out the coordinates of the intercept (Q1)	• Check your answer for the coordinates of the y-axis intercept. You may want to substitute the value for x at this point into the equation.
Substitution incorrectly used when figuring out the coordinates of the intercept (Q1)	• Check your answer for the intercept by using the Pythagorean rule.
Cannot get started (Q2)	• Try sketching a circle with these features. Write what you know on your sketch. • Try out an equation. Check to see if it fits the conditions.
Graph correctly sketched but equation is incorrect (Q2) For example: $(x - 2)^2 + (y - 6)^2 = 36$.	• Check your equation is correct by figuring out the coordinates of the x-axis intercepts and y-axis intercepts.
Little or no explanation provided (Q2)	• Sketch your circle and write the coordinates of its center and its radius.
Correctly answers all the questions The student needs an extension task.	• In Question 2, suppose the coordinates of the intercepts of the circle are always integers. How many circles could you draw? Explain your answer.

SUGGESTED LESSON OUTLINE

Whole-class introduction (20 minutes)

Give each student a mini-whiteboard, pen, and eraser.

Show Slide P-1 of the projector resource:



Each of these three circles might be considered to be the odd one out.
Choose one and write on your mini-whiteboard why it might be the odd one out.
What properties have the other two circles got in common that the third does not have?

Ask students for justifications. As they do this, write the properties they mention on the board. You may need to encourage students to think about the properties of the intercepts of each circle. There may be more than one reason for each.

- Graphs B and C have the same radius. Graph A has a different radius.
- Graphs A and C have center $(6,-4)$. Graph B has center $(-4, 2)$.
- Graphs A and B have two x-axis intercepts. Graph C only has one x-axis intercept.
- Graphs A and B have one y-axis intercept. Graph C has no y-axis intercept.

Show Slide P-2 of the projector resource:

Odd One Out

A. $(x + 4)^2 + (y - 3)^2 = 16$

B. $(x - 6)^2 + (y - 7)^2 = 25$

C. $(x - 5)^2 + y^2 = 25$

Ask students to repeat the exercise with the three equations, again looking at centers, radii and intercepts. These will take a bit more thought, so allow students time. If students are struggling, encourage them to discuss the problem with a neighbor before asking one or two students for an explanation. Encourage the rest of the class to query explanations.

- Equations B and C have a radius of 5. Equation A has a radius of 4.
- Equations A and C have two x-axis intercepts. Equation B has no x-axis intercept.
- Equations A and C have one y-axis intercept. Equation B has no y-axis intercept.

- Equation C is the odd one out as its center is the only one on the x-axis.

Students who struggle may be encouraged to either (i) substitute $x = 0$, or $y = 0$ into each equation, or (ii) sketch a graph using the key features and the Pythagorean theorem:

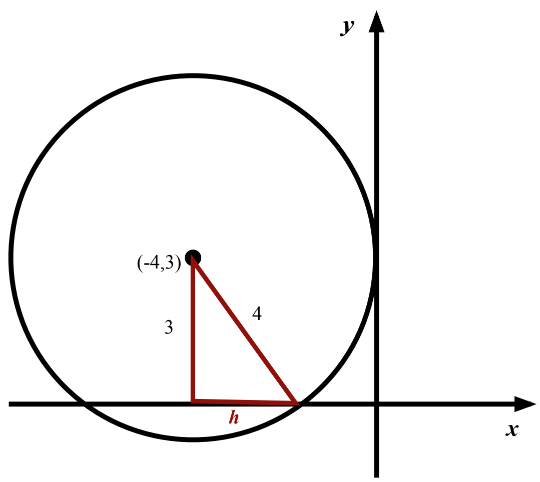
(i) Equation A: $(x + 4)^2 + (y - 3)^2 = 16$.

When $x = 0$ then
$$4^2 + (y - 3)^2 = 16$$
$$(y - 3)^2 = 0$$
$y = 3$. There is one y-axis intercept.

When $y = 0$ then
$$(x + 4)^2 + (-3)^2 = 16$$
$$(x + 4)^2 + 9 = 16$$
$$(x + 4)^2 = 7$$

There are two x-axis intercepts: $x = -4 + \sqrt{7}$ and $x = -4 - \sqrt{7}$.

(ii)



$h^2 = 4^2 - 3^2 = 7$

$h = \sqrt{7}$

Therefore, the x-axis intercepts are at $(-4 + \sqrt{7}, 0)$ and $(-4 - \sqrt{7}, 0)$.

Collaborative activity 1 (20 minutes)

Organize the class into groups of two or three students. Give each group the *Table Headings: Categorizing Equations*, *Card Set: Equations 1*, and a large sheet of paper.

Show Slide P-3 of the projector resource and ask students to use the *Table Headings* to create the table on their poster. The table should cover the entire poster.

Categorizing Equations

	No x-axis intercept	One x-axis intercept	Two x-axis intercepts
No y-axis intercept			
One y-axis intercept			
Two y-axis intercepts			

Slide P-4 of the projector resource summarizes how students are to work collaboratively.

> *You are now going to continue to sort equations of circles according to their number of x-axis intercepts and y-axis intercepts.*

Working Together

1. Take turns to place the equation cards in one of the categories in the table.
   - To do this, figure out the coordinates of any intercepts.
   - You may want to sketch the graph of the equation.
2. If you place a card, explain how you came to your decision.
3. Your partner should check your answer using a different method.
4. You **all** need to be able to agree on and explain the placement of every card.
5. Write some additional information or include a drawing as part of your explanation.
6. You are to ask each other for help before asking the teacher.

> *Once you have placed all four cards, figure out what the equations have in common.*

The purpose of this structured group work is to make students engage with each others' explanations, and take responsibility for each other's understanding.

Encourage students not to rush into the activity, but spend some time thinking about how they can approach the task.

> *What information can you get from the cards?*
>
> *How can you use this information?*

Encourage students to spend time justifying fully the position of each card.

While students work in small groups you have two tasks: to note different student approaches to the task and support student reasoning.

Note different student approaches to the task

Notice how students make a start on the task, any interesting ways of explaining a categorization, any attempts to generalize, where they get stuck, and how they respond if they do come to a halt. For example, do students plot the graph of an equation accurately? Do students use the Pythagorean theorem or substitution to figure out the coordinates of the intercepts on the circle? What do students do when the x or y coordinate of an intercept is imaginary? Do students sketch a graph of the equation? If so, when do they sketch a graph: at the start to figure out the number of intercepts or after they have figured out the coordinates of the intercepts? Do students notice that the equations in a column or row have a common feature?

You can use this information to focus the whole-class discussion.

Support student reasoning

Try not to make suggestions that move students towards particular categorization, instead draw student's attention to critical mathematical features that they might not yet understand. Try to create a conjecturing atmosphere by asking questions to help students to reason together. You may decide to use some of the questions and prompts from the *Common issues* table.

If a student struggles to get started, encourage them to ask you or their partner a specific question about the task. Articulating the problem in this way can sometimes offer a direction to pursue that was previously overlooked.

Some students may try to accurately plot the graph of the equation. Encourage them to sketch the graph instead.

What does the equation of the circle tell you about the center of the circle and the radius?

Can you use this information to sketch a circle?

When students substitute $x = 0$, or $y = 0$ into the equations, some may have difficulty manipulating it. For example, students may write:

$(x + 3)^2 + 6^2 = 25$

$x + 3 + 6 = 5$

$x = -4$.

In which case you may decide to ask students to check their answers. The formative assessment lesson *Evaluating Statements about Radicals* addresses this misconception.

If you find one student has placed a card in a particular category on the table, ask another student in the group to provide an explanation.

Chan placed this card here. Cheryl, why does Chan think this equation has this number of intercepts?

Can you show me a different method to the one Chan used?

How can you check that the number of intercepts for this equation is correct?

If you find the student is unable to answer these questions, ask them to discuss the work further. Explain that you will return in a few minutes to ask a similar question.

Sharing posters (15 minutes)

As students finish categorizing the first four cards, ask them to check their work with that of a neighboring group.

Slide P-5 of the projector resource summarizes how students should work.

Sharing Posters

1. Check to see which equations have been placed in different categories from your own.
2. If there are differences, ask for an explanation. If you still don't agree, explain your own thinking.
3. Once there is agreement, look to see what these equations or their graphs have in common.
   - What is it about these equations that give them this common graphical feature?
   - Can you use this information to think of a quick way of figuring out the number of intercepts for a particular equation?

Encourage students to focus on the structure of the equations and start to make generalizations.

What do all four sketches have in common? [No x-axis intercept.]

What is it about the equations that give their graphs this common feature?

What is it about the equation that means the graph has no, one or two intercepts?

Tell me about the relationship between the coordinates of the center of a circle and its radius for each circle. [Suppose (m,n) is the center of a circle and r its radius. For no x-axis intercept $|n| >$

radius. For no y-axis intercept, |m| > radius. For one y-axis intercept, |m| = radius. For two y-axis intercepts, |m| < radius, and so on.]

Collaborative activity 2 (20 minutes)

After students have checked each other's work and started to think about generalizations, ask them to return to their original groups.

Give each group *Card Set: Equations 2*.

Students are now to categorize the remaining cards. They can either continue figuring out the values of the *x*-axis and *y*-axis intercepts for each equation or they can use the structure of the equation to categorize it.

If there is no equation on the card they are to add one that fits the sketch. More than one equation fits the sketch for Cards 8, 9 and 10. Card 10 should be placed in the empty cell on the poster. If students struggle to figure out the equations for these cards, you may want to ask:

Look at the other cards in this row/column. What do they have in common?
What is it about the graphs that give them this common feature?
What is it about the equations of these graphs that give them this common feature?

Again listen carefully to students' explanations and note any difficulties they encounter. You can use this information in the whole-class discussion at the end of the lesson.

Support students as in the first collaborative activity.

As groups finish categorizing the cards, give them a glue stick. They are to glue all the cards onto the paper and then attach the poster to the classroom wall for everyone to see.

Whole-class discussion (15 minutes)

During the discussion you may want to use Slide P-3 of the projector resource and transparencies of *Equations 1* and *Equations 2*.

Categorizing Equations

	No *x*-axis intercept	One *x*-axis intercept	Two *x*-axis intercepts
No *y*-axis intercept			
One *y*-axis intercept			
Two *y*-axis intercepts			

Organize a discussion about what has been learned. The intention is that you work with the students to make the math of the lesson explicit and encourage them to make generalizations.

First select a categorization that most groups placed correctly. This approach may encourage good explanations. Then select one or two cards that most groups found difficult to categorize.

Once one group has justified their choice for a particular categorization, ask other students to contribute ideas of alternative approaches and their views on which reasoning method was easier to follow. The intention is that you focus on getting students to understand and share their **reasoning**, not just checking that everyone produced the right answers. Use your knowledge of the students' individual and group work to call on a wide range of students for contributions. You may want to draw on the questions in the *Common issues* table to support your own questioning.

Michael, where did you place this card? How did you decide? Why does that card not go in this category?
Does anyone have any questions about Michael's method?
Can someone else, put Michael's method into your own words?
Did anyone use a different/similar method?

If you want to explore, in a more general way, when x-axis and y-axis intercepts occur, write on the board:

$$(x-m)^2 + (y-n)^2 = \text{radius}^2$$

And ask students:

What do you notice about equations that have no y-axis intercepts? [$|m| >$ radius.]

What do you notice about equations that have just one y-axis intercept? [$|m| =$ radius.]

What do you notice about equations that have two y-axis intercepts? [$|m| <$ radius.]

Follow-up lesson: reviewing the assessment task (20 minutes)

Return to students their original assessment *Going Round in Circles 2*.

If you have not added questions to individual pieces of work or highlighted questions on a list of questions, then write your list of questions on the board. Students should select from this list only the questions they think are appropriate to their own work.

If students struggled with the original assessment, they may benefit from revising this assessment. In order that students can see their own progress, ask them to complete the task using a different color pen or give them a second blank copy of the task. Otherwise give students a copy of the task *Going Round in Circles 2 (revisited)*.

Read through your original solutions to the task.

Use what you have learned in the lesson to answer the questions.

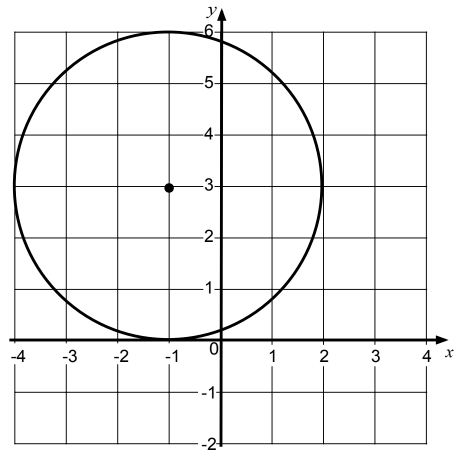
Then complete the new assessment task/revise your answers.

SOME TEACHERS GIVE THIS AS A HOMEWORK TASK.

SOLUTIONS

Assessment task: *Going Round in Circles 2*

1. The graph shows there is one x-axis intercept at $(-1,0)$.



Students may use substitution or the Pythagorean theorem to figure out the y-axis intercepts for the graph of the equation $(x+1)^2+(y-3)^2=9$.

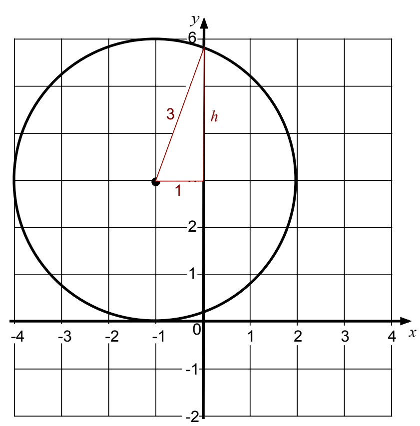
Using substitution:

When $x=0$ then $1^2+(y-3)^2=9$; $(y-3)^2=8$

$$y = 3+\sqrt{8} = 3+2\sqrt{2} \quad \text{or} \quad y = 3-\sqrt{8} = 3-2\sqrt{2}$$

Therefore, the y-axis intercepts are at $\left(0,\ 3+2\sqrt{2}\right)$ and $\left(0,\ 3-2\sqrt{2}\right)$.

Using the Pythagorean theorem:



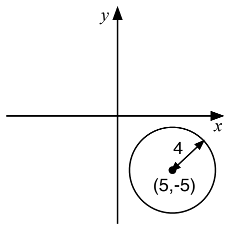
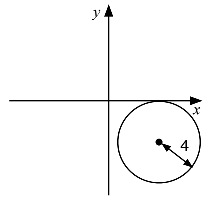
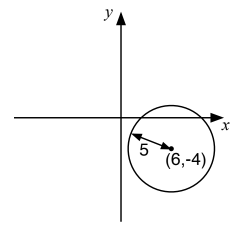
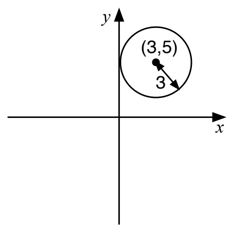
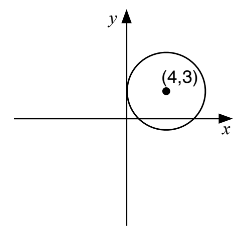
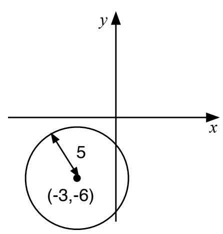
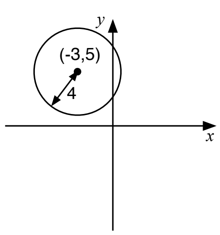
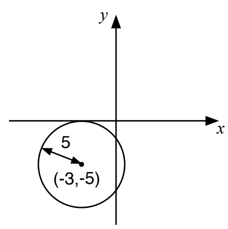
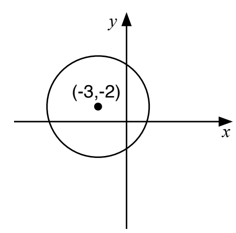
$$h^2 = 3^2 - 1^2 = 8; \quad h = \sqrt{8} = 2\sqrt{2}$$

$$y = 3+2\sqrt{2} \quad \text{or} \quad y = 3-2\sqrt{2}$$

Therefore, the y-axis intercepts are at $\left(0,\ 3+2\sqrt{2}\right)$ and $\left(0,\ 3-2\sqrt{2}\right)$.

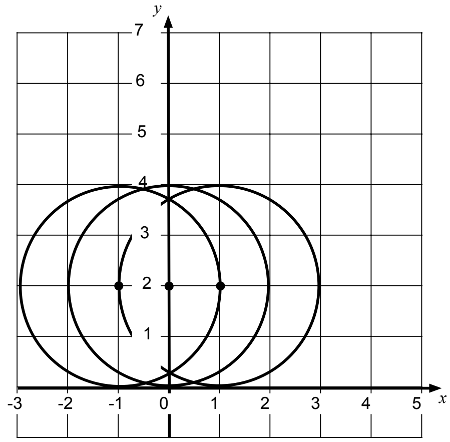
2. Students may provide any examples in the form:
$(x-6)^2+(y-m)^2=36$, or $(x+6)^2+(y-m)^2=36$ or
$(x-6)^2+(y+m)^2=36$, or $(x+6)^2+(y+m)^2=36$ where $|m|<6$.

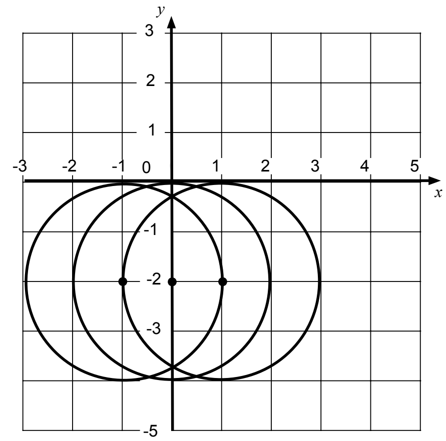
Collaborative activity: *Making Posters*

	No x-axis intercept	One x-axis intercept	Two x-axis intercepts
No y-axis intercept	3. $(x - 5)^2 + (y + 5)^2 = 16$ 	8. The equation is: $(x - m)^2 + (y + 4)^2 = 16$ where $m > 4$ 	5. $(x - 6)^2 + (y + 4)^2 = 25$ x-axis intercepts at (3, 0) and (9, 0) 
One y-axis intercept	1. $(x - 3)^2 + (y - 5)^2 = 9$ y-axis intercept at (0,5) 	10. The equation is: $(x - r)^2 + (y - r)^2 = r^2$ where $r = radius$. x-axis intercept at $(r,0)$ y-axis intercept at $(0,r)$	7. The equation is: $(x - 4)^2 + (y - 3)^2 = 16$ y-axis intercept at (0,3) x-axis intercepts are at $\left(4+\sqrt{7},\ 0\right)$ and $\left(4-\sqrt{7},\ 0\right)$. 
Two y-axis intercepts	2. $(x + 3)^2 + (y + 6)^2 = 25$ y-axis intercepts at (0, –2) and (0. –10)  4. $(x + 3)^2 + (5 - y)^2 = 16$ y-axis intercepts are at $\left(0,\ 5+\sqrt{7}\right)$ and $\left(0,\ 5-\sqrt{7}\right)$. 	6. $(y + 5)^2 + (3 + x)^2 = 25$ x-axis intercept (–3, 0) y-axis intercepts (0, –1) and (0, –9) 	9. The equation is: $(x + 3)^2 + (y + 2)^2 = r^2$ where $r > 3$. 

Assessment task: *Going Round in Circles 2 (revisited)*

1. There are six possible graphs that fit the criteria:





The equation should take the form:

$$(x - \text{m})^2 + (y - 2)^2 = 4 \text{ or } (x - \text{m})^2 + (y + 2)^2 = 4, \text{ where m} = -1, 0, \text{ or } 1.$$

2. Students may use substitution or the Pythagorean theorem to figure out the y-intercepts for the graph of the equation $(x - 4)^2 + y^2 = 36$.

Using substitution:

When $x = 0$ then $16 + y^2 = 36$

$y^2 = 20$

$y = \sqrt{20}$ or $y = -\sqrt{20}$

$y = 2\sqrt{5}$ or $y = -2\sqrt{5}$

Therefore, the y-axis intercepts are at $\left(0, 2\sqrt{5}\right)$ and $\left(0, -2\sqrt{5}\right)$.

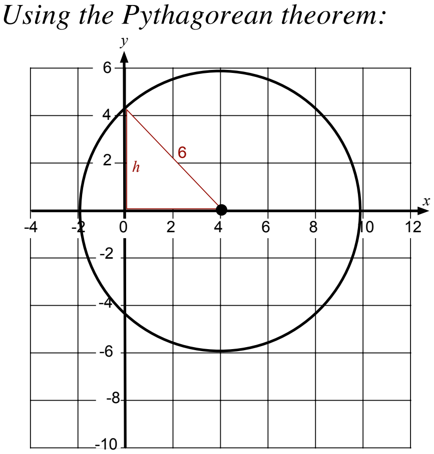
$y = 0$ then $(x - 4)^2 = 36$

$(x - 4)^2 = 36$

$x = 4 + 6 = 10$ or $x = 4 - 6 = -2$

Therefore the x-axis intercepts are at $(10, 0)$ and $(-2, 0)$.

Using the Pythagorean theorem:



$h^2 = 6^2 - 4^2 = 20$

$y = 2\sqrt{5}$ OR $y = -2\sqrt{5}$

Therefore, the y-axis intercepts are at:

$\left(0, 2\sqrt{5}\right)$ and $\left(0, -2\sqrt{5}\right)$.

Going Round in Circles 2

1. Complete the drawing of the graph of the equation $(x + 1)^2 + (y - 3)^2 = 9$ to show the x- axis and y-axis. Add numbers to each of these axes.

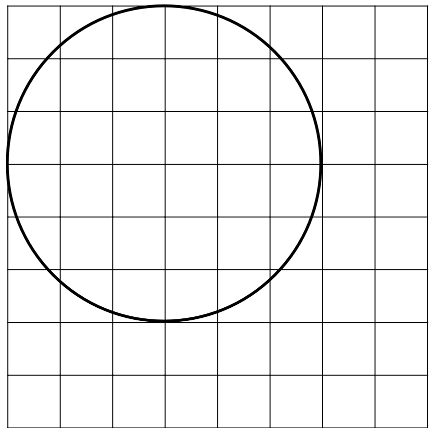


Figure out the co-ordinates of any x-intercepts and y-intercepts.

Explain your answer(s).

2. Write an equation of a circle that has two x-intercepts but just one y-intercept and a radius of 6.

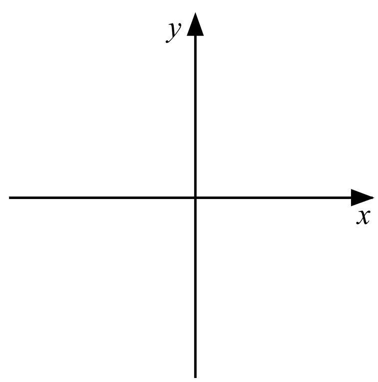
Explain your answer.

Table Headings: Categorizing Equations

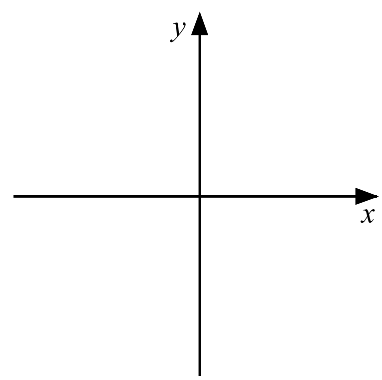
No x-axis intercept	One x-axis intercept
Two x-axis intercepts	No y-axis intercept
One y-axis intercept	Two y-axis intercepts

Card Set: Equations 1

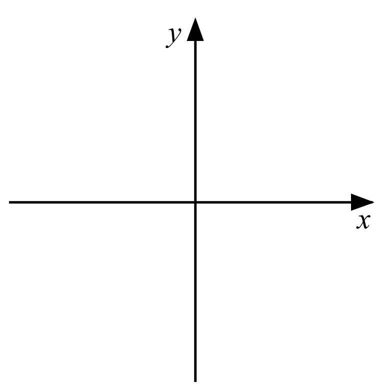
1. $(x - 3)^2 + (y - 5)^2 = 9$



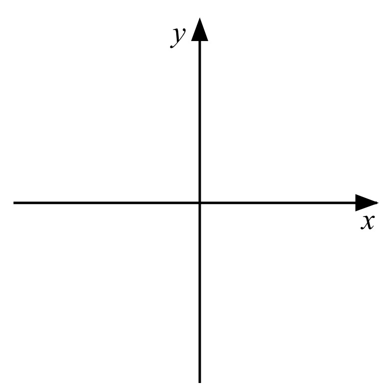
2. $(x + 3)^2 + (y + 6)^2 = 25$



3. $(x - 5)^2 + (y + 5)^2 = 16$

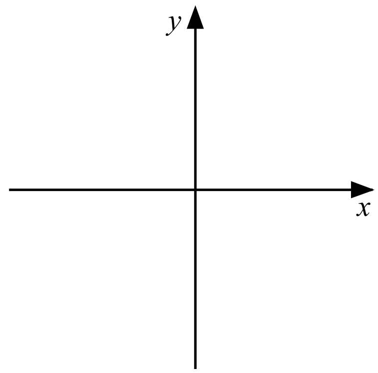


4. $(x + 3)^2 + (5 - y)^2 = 16$

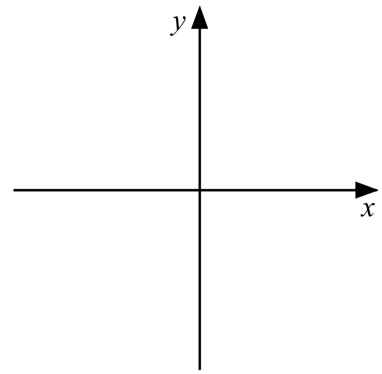


Card Set: Equations 2

5. $(x - 6)^2 + (y + 4)^2 = 25$



6. $(y + 5)^2 + (3 + x)^2 = 25$



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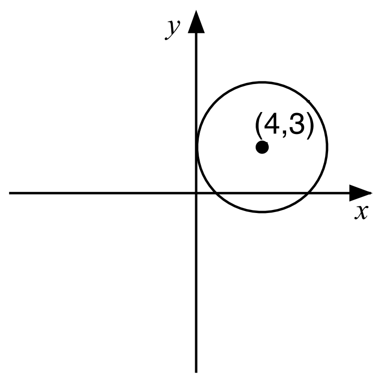

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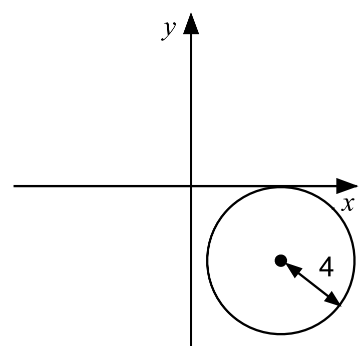

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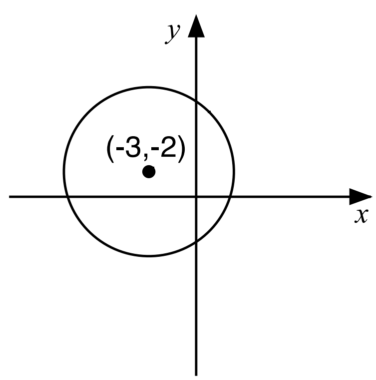
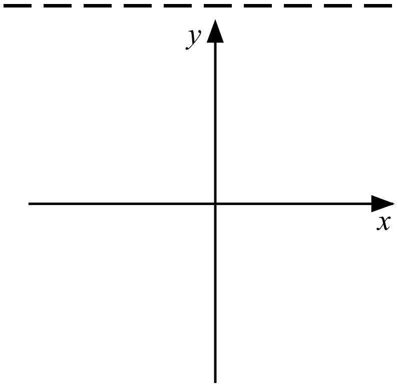


10.



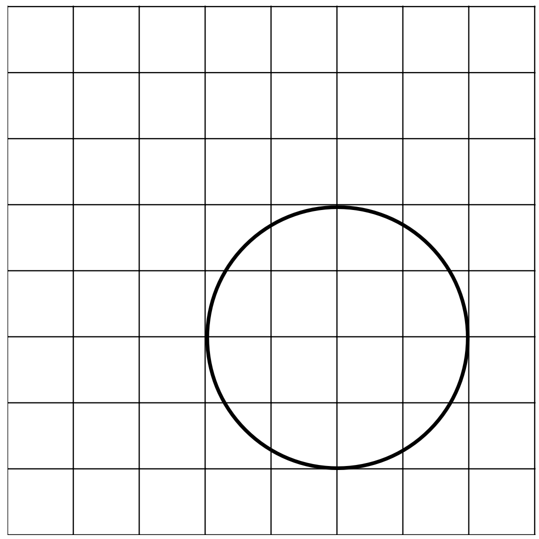
Categorizing Equations

	No x-axis intercept	One x-axis intercept	Two x-axis intercepts
No y-axis intercept			
One y-axis intercept			
Two y-axis intercepts			

Going Round in Circles 2 (revisited)

1. A circle has one x-intercept, two y-intercepts and a radius of 2.
   The coordinates of the center are integers.

   Draw the x- axis and y-axis on the grid below to show a possible graph of this circle.
   Add numbers to each axis.

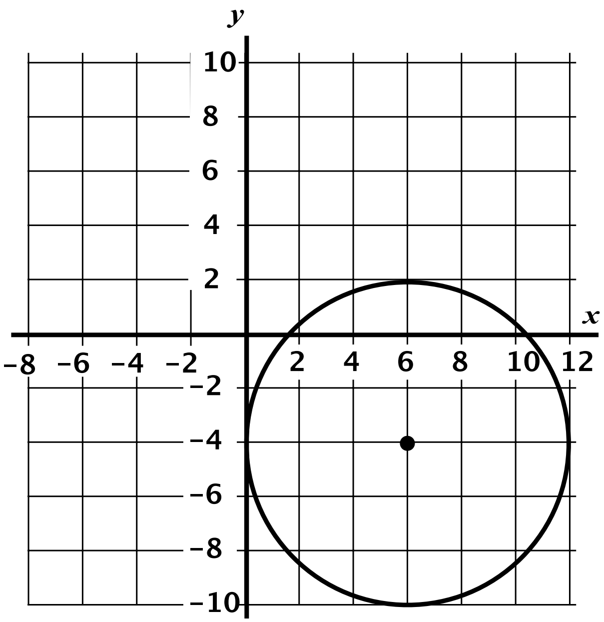


   What is the equation of your graph?
   Explain your answer.

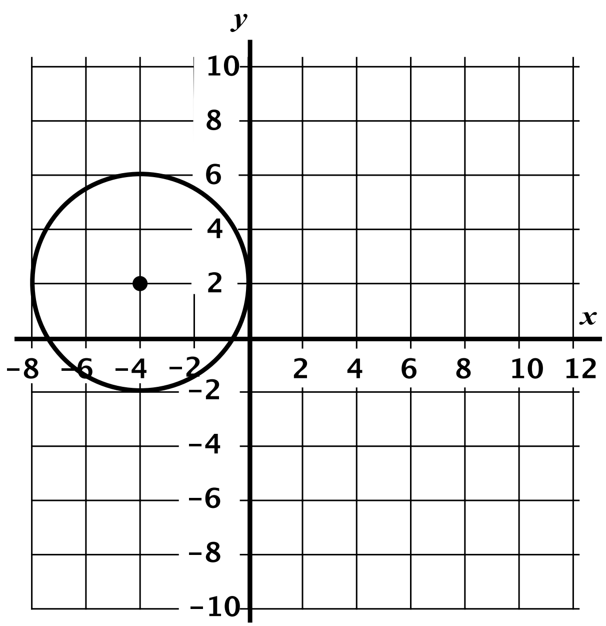
2. Figure out the co-ordinates of any x-intercepts and y-intercepts for the graph $(x - 4)^2 + y^2 = 36$.
   Explain your answer.

Odd One Out

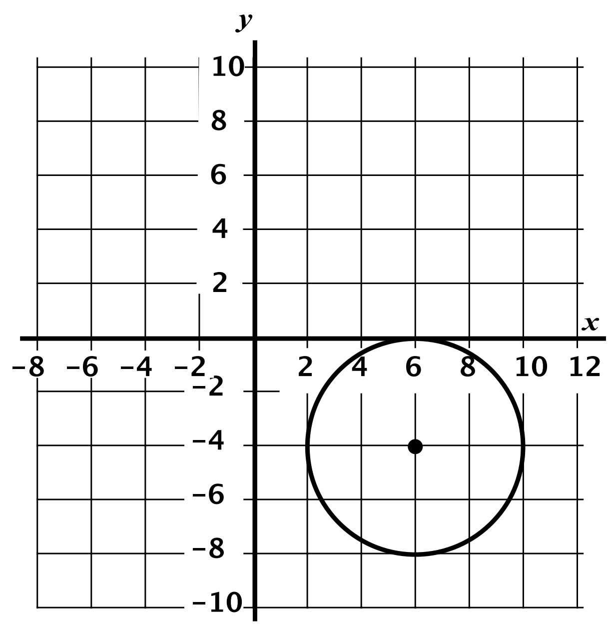
Graph A



Graph B



Graph C



Odd One Out

A. $(x + 4)^2 + (y - 3)^2 = 16$

B. $(x - 6)^2 + (y - 7)^2 = 25$

C. $(x - 5)^2 + y^2 = 25$

Categorizing Equations

	No x-axis intercept	One x-axis intercept	Two x-axis intercepts
No y-axis intercept			
One y-axis intercept			
Two y-axis intercepts			

Working Together

1. Take turns to place the equation cards in one of the categories in the table.
   - To do this, figure out the coordinates of any intercepts.
   - You may want to sketch the graph of the equation.

2. If you place a card, explain how you came to your decision.

3. Your partner should check your answer using a different method.

4. You **all** need to be able to agree on and explain the placement of every card.

5. Write some additional information or include a drawing as part of your explanation.

6. You are to ask each other for help before asking the teacher.

Sharing Posters

1. Check to see which equations have been placed in different categories from your own.

2. If there are differences, ask for an explanation. If you still don't agree, explain your own thinking.

3. Once there is agreement, look to see what these equations or their graphs have in common.
   - What is it about these equations that give them this common graphical feature?
   - Can you use this information to think of a quick way of figuring out the number of intercepts for a particular equation?

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
Shell Center Team at the Center for Research in Mathematical Education
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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org