CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Representing 3D Objects in 2D

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Representing 3D Objects in 2D

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to visualize two-dimensional cross-sections of representations of 3-dimensional objects. In particular, the lesson will help you identify and help students who have difficulties recognizing and drawing 2-dimensional cross-sections at different points along a plane of a representation of a 3-dimensional object.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

G-GMD: Visualize relationships between two-dimensional and three-dimensional objects.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 5 and 7:

1. Make sense of problems and persevere in solving them.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.

INTRODUCTION

- Before the lesson, students work individually on an assessment task designed to reveal their current understanding and difficulties. You review their solutions and create questions for students to consider, to help improve their work.
- During the lesson, students work in groups on a collaborative activity. They match representations of three-dimensional objects with two-dimensional cross-sections. Students then share their work with another group.
- In a whole-class discussion, students compare and evaluate the methods they have seen and used.
- In a follow-up lesson, students review their initial solutions and then use what they have learned to either revise the same introductory assessment task or complete a different task.

MATERIALS REQUIRED

- Each student will need a copy of the two assessment tasks, *Vessels of Water* and *Vessels of Water (revisited),* a mini-whiteboard, pen, and eraser.
- Each small group of students will need the cut-up cards *Flowing Water* and *Shape of the Surface of the Water (1)* and *(2)*, a large sheet of paper for making a poster, and a glue stick.
- There is a projector resource to support whole-class discussions.

For the introduction, a real example, such as soda bottles filled to different levels with colored water, may help students visualize the two-dimensional cross-sections.

In the collaborative activity students may also benefit from seeing real examples of the three-dimensional objects such as cones, hemispheres, cuboids, triangular prisms, and rectangular pyramids.

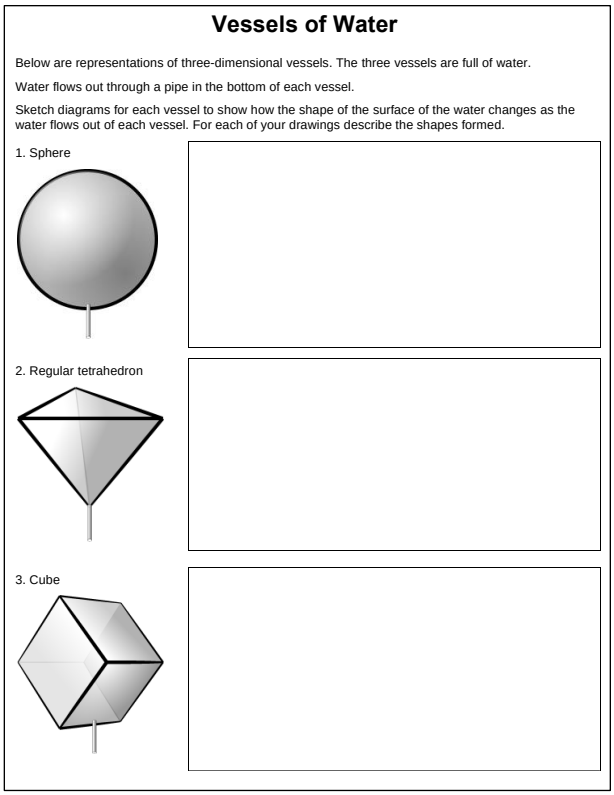
TIME NEEDED

20 minutes before the lesson, a 75-minute lesson (or split into two shorter ones), and 20 minutes in a follow-up lesson. Timings are approximate and will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *Vessels of Water* (20 minutes)

Have the students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the next lesson.



Give each student a copy of the task *Vessels of Water*.

> *Read through the questions and try to answer them as carefully as you can.*
>
> *Imagine looking down on the vessel. Your sketches should represent the shape of the surface of the water as water flows out of the vessel.*

It is important that students are allowed to answer the questions without your assistance, as far as possible.

Students should not worry too much if they cannot understand or do everything, because in the next lesson they will engage in a similar task, which should help them. Explain to students that by the end of the next lesson, they should expect to answer questions such as these confidently. This is their goal.

Assessing students' responses

Collect students' responses to the task and note what their work reveals about their current levels of understanding and their different approaches.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores and will distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a list of questions. Some suggestions for these are given in the *Common issues* table on the next page. We suggest that you make a list of your own questions, based on your students' work, using the ideas on the following page. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these questions on the board when you return the work to the students in the follow-up lesson.

Common issues:	Suggested questions and prompts:
No drawings of the shape of the surface of the water	• Now provide some diagrams to show the shape of the surface of the water as the water level changes. • What is the shape of the surface of the water when the vessel is almost full/empty/half full? Draw these shapes.
Confusion between two-dimensional representation of the shape and two-dimensional representation of the surface of the water For example: The student draws two concentric circles, one representing the shape and the other representing the surface area of the water (Q1). Or: The student draws several vertical cross-sections of the sphere, each with different water levels (Q1).	• Your drawings do not need to include the shape of the vessel, just the shape of the surface of the water. • Imagine looking down on the vessel as the water flows out of it. Sketch the shape of the surface of the water.
Description lacks precision For example: The student does not state the radius of the largest circle or when this occurs (Q1). Or: The student does not state that the triangles will be equilateral as the level of the water changes (Q2). Or: The student does not state how the dimensions of the rectangle change as the level of the water changes (Q3).	• What is the radius of the largest circle? When does this occur? • What can you say about the properties of the triangles? • How do the dimensions of the rectangle change? • What are the dimensions of the biggest rectangle? When does this occur?
The drawn shapes lacks precision For example: The student draws a series of congruent triangles (Q2). Or: The student draws rectangles that show the width **and** length changing as the level of the water flows out of the vessel (Q3).	• Imagine looking down on the tetrahedron/cube when it is three quarters full, then when it is half full. What has changed about the shape of the surface of the water? • What are the dimensions of the shape when the cube/sphere is half full?
Difficulty representing the middle section of the cube (Q3)	• Imagine the vessel is half full of water. What is the shape of the surface of the water? • What can you say about the dimensions of this shape?
Assumption that because the shape is a cube the shape of the surface of the water will be squares (Q3)	• What determines the width and length of the shape of the surface of the water?

SUGGESTED LESSON OUTLINE

Whole-class introduction (15 minutes)

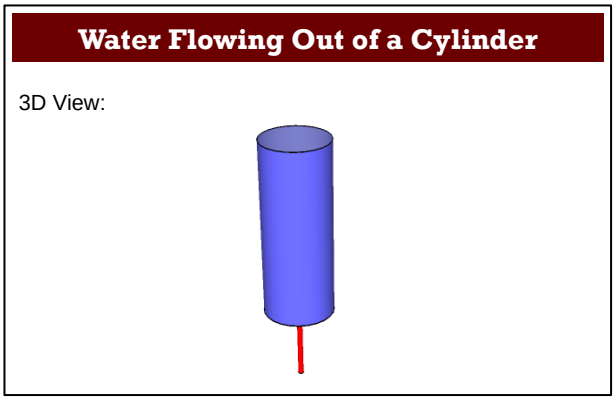
This introduction will provide students with a model of how they should justify their matches in the collaborative activity.

If available, use cylinders filled to different levels with colored water to help students visualize the two-dimensional cross-sections.

Ask questions that provoke thoughtful answers and treat these answers as building blocks for further dialogue, rather than end points in the discussion. When asking students to explain their work, select examples that will promote interesting discussions. If students are struggling, encourage them to discuss the task with a partner.

Give each student a mini-whiteboard, pen, and eraser.

Display Slide P-1 of the projector resource:

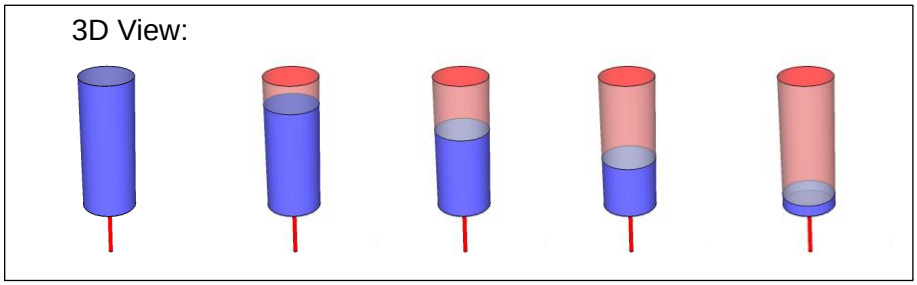


The cylinder is full of water. This water flows out through a pipe at the bottom of the cylinder.

Imagine looking down on the cylinder as the water flows out of it.

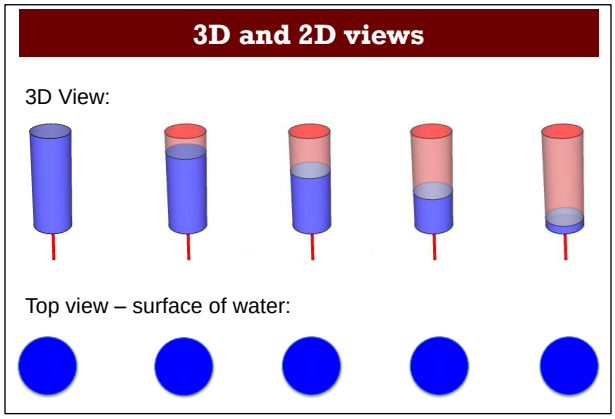
Draw the shape of the surface of the water at five different levels.

Then show the sequence of Slides P-2 to P-5 of the projector resource. This visualization of the flow of liquid out of the cylinder should help students to understand the context.

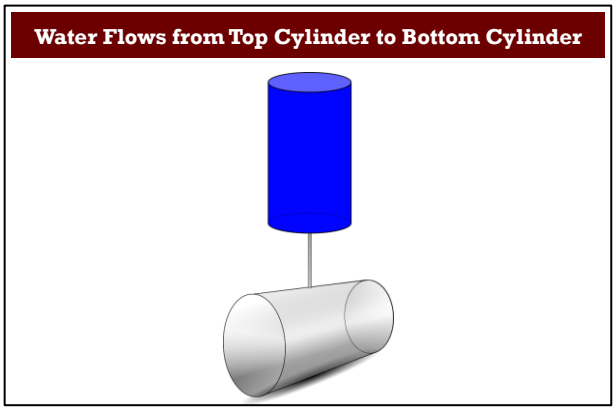


After about a minute, ask students to show you their whiteboards. If there is a range of different drawings, ask students to justify their answers. Encourage the rest of the class to challenge their explanations.

Confirm students' drawings by showing Slide P-6 of the projector resource:

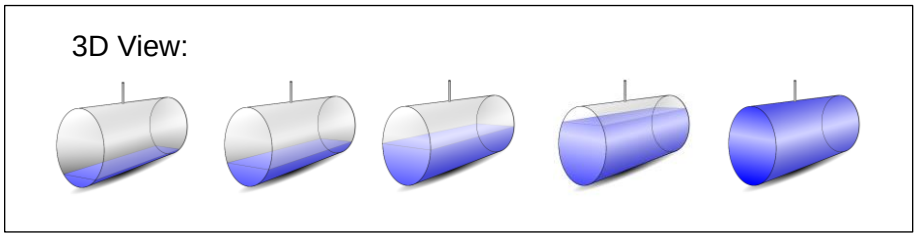


Display Slide P-7 of the projector resource:



The water flows from the top cylinder into another cylinder that is at a different orientation.

Then show the sequence of Slides P-8 to P-12 of the projector resource. This visualization of the flow of liquid into the cylinder should help students to understand the context.



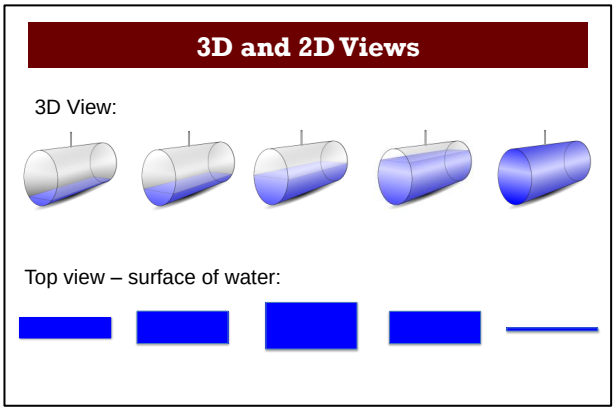
Ask students to draw the shape of the surface of the water in the bottom cylinder, at five different levels.

After a few minutes, ask a couple of students with different drawings to justify them.

Again, imagine looking down on the bottom cylinder.
As the bottom cylinder fills with water, what is the shape of the surface of the water?

Will the width and length of this shape change? [The length of the rectangle will remain the same. The width will increase until it is the same as the diameter of the cylinder, it will then decrease.]

Confirm students' drawings by showing Slide P-13:



Collaborative activity (30 minutes)

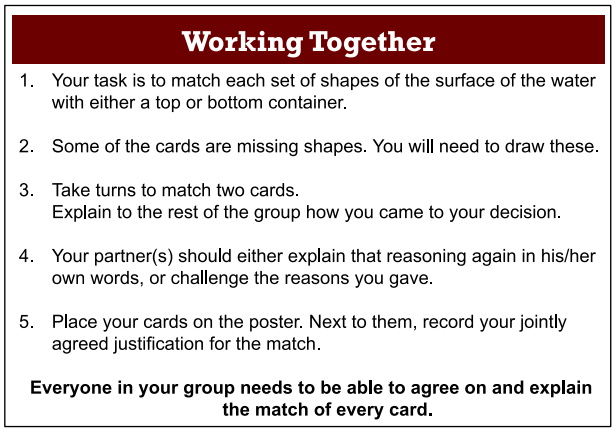
Organize the class into groups of two or three students.

Give each group the cut-up cards *Flowing Water, Shape of the Surface of the Water 1* and *2*, and a large sheet of paper for making a poster. If possible, give each group a selection of real examples of three-dimensional shapes.

First explain the task:

> *Look at the three-dimensional diagrams I've given out. Each one shows an arrangement of two* ***identical*** *containers connected by a vertical pipe.*
>
> *The top container starts out full of water. This water flows into the bottom container.*
>
> *Imagine what the shape of the surface of the water will look like in the top container and in the bottom.*
>
> *Imagine how these shapes will change as the water flows from the top container into the bottom.*
>
> *The two-dimensional diagrams show the possible shapes of the surface of the water, at different times. The first shape in each diagram is* ***either*** *at the top of the shape for the top container* ***or*** *the bottom of the shape for the bottom container.*

Then explain how students are to work collaboratively. Slide P-14 of the projector resource summarizes how students should work together:



The purpose of this structured group work is to encourage students to engage with each other's explanations and take responsibility for each other's understanding.

It does not matter if students do not manage to match all of the cards. It is more important that everyone in each group understands the reasons for each match.

Encourage students to mark on the three-dimensional containers the approximate position of each two-dimensional shape.

While students work in small groups you have two tasks, to note different student approaches to the task and to support student reasoning.

Note different student approaches to the task

Notice how students make a start on the task, where they get stuck, and how they respond if they do come to a halt. You can use this information to focus a whole-class discussion at the end of the lesson. Do students assume that the two-dimensional shapes will have a similar shape to one of the faces of the three-dimensional container? Do students notice if one dimension of a shape remains the same as another dimension changes? Do students use symmetry of the three-dimensional containers or two-dimensional shapes? Do students compare the shapes for the top container with the shapes for a bottom container?

Support student reasoning

Try not to make suggestions that move students towards a particular match. Instead, ask questions to help students to reason together. You may want to use the *Common issues* to support your own questioning.

> *What is the shape of the surface of the water at the start/end?*
>
> *When will the surface area of the water be biggest/smallest? Show me on the container.*
> *What will be the shape of the surface of the water?*
>
> *Do all dimensions of the shape of the surface of the water change as the level of the water in the container changes?*
>
> *Could any of the shapes for the surface of the water in the top container be congruent to a shape for the surface of the water in the bottom container?*
>
> *What are the differences between these two diagrams? [Select pairs of cards that are similar, for example S2 and S8, or S4 and S7, or S1 and S6.]*
> *How will these differences affect the shape of the container?*
>
> *How will the shape of the water surface differ if the container is filled rather than drained of water?*

Encourage students to draw the missing shapes accurately:

> *You have drawn in this shape for the surface of the water.*
> *Where do you think the water level is on the container?*

If you find one student has matched two cards, challenge another student in the group to provide an explanation:

> *Jordan matched these cards. Kylie, why does Jordan think these two cards go together?*

If you find the student is unable to answer this question, ask them to discuss the work further. Explain that you will return in a few minutes to ask a similar question.

If the whole-class is struggling on the same issue, you could write a couple of questions on the board and hold a whole-class discussion.

Sharing work (15 minutes)

As students finish matching the cards, ask one student from each group to visit another group's desk.

If you are staying at your desk, be ready to explain the reasons for your group's matches.

If you are visiting another group, write your card matches on a piece of paper.
Go to another group's desk and check to see which matches are different from your own.

If there are differences, ask for an explanation. If you still don't agree, explain your own thinking.

When you return to your own desk, you need to consider, as a group, whether to make any changes to your own work.

You may want to use Slide P-15, *Sharing Work*, to display these instructions.

As students finish sharing their work, give them a glue stick.

Glue your cards to the poster.

If you do revise your work then write on the poster the reason for your revision.

Extension Activity

If students finish quickly, give them a sheet of paper and ask them to sketch their own containers, consisting of two vessels connected by a vertical pipe. They are to give their drawings to a partner and ask them to sketch the surface area of the water as the water drains out of the top container and into the bottom one. Then students are to check the work together.

Whole-class discussion (15 minutes)

Ask students to display their posters at the front of the room for all to see. Slides P-16 to P-18 show the card sets, which may help with this discussion.

Organize a whole-class discussion about the different strategies used to match the cards. You may first want to select a pair of cards that most groups matched correctly. This approach may encourage good explanations. Then select one or two cards that most groups found difficult to match.

Once one group has justified their choice for a particular match, ask other students to contribute ideas of alternative approaches, and their views on which reasoning method was easier to follow. The intention is that you focus on getting students to understand and share their **reasoning**, not just checking that everyone produced the right answers.

You may want to ask a selection of these questions:

Which was the easiest card to match? Why? What strategy did you use?
Does anyone disagree with the match?

Which was the most difficult card to match? Why? What strategy did you use?
Does anyone disagree with the match?

When sharing your work with another group, did any of you see anything to make you change your mind? Why?

What strategy did you use to match these cards? Does anyone have a different strategy? Does anyone have a strategy that works on more than one card?

How did you decide to fill in the missing shapes on this card?

What patterns seemed to emerge as you worked?

Make specific reference to issues and misconceptions you have noticed during the collaborative activity. You may want to draw on the questions in the *Common issues* table to support your own questioning.

Follow-up lesson: review of initial assessment task (20 minutes)

Return the original assessment, *Vessels of Water*, to the students.

If you have not added questions to individual pieces of work, or highlighted questions on a sheet, then write your list of questions on the board. Students should select from this list, only the questions they think are appropriate to their own work.

Read through your original solutions to the task and think about what you have learned this lesson.

Carefully read through the questions I have written.

Spend a few minutes answering these questions

You may want to make notes on your mini-whiteboard.

If students struggled with the original assessment then they may benefit from revising this assessment. In order that students can see their own progress, ask them to complete the task using a different color pen. Otherwise give students a copy of the task *Vessels of Water (revisited.)*

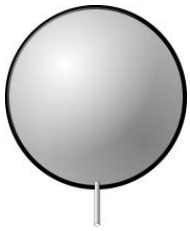
Use what you have learned to complete the new assessment task/revise your answers.

If you find you are running out of time, then you could give this as a homework task.

SOLUTIONS

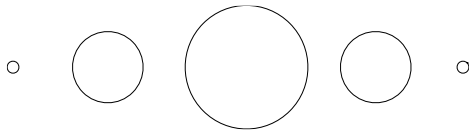
Assessment task: *Vessels of Water*

1.



As the level of water decreases in the sphere, the shape of the surface of the water will always be circular. The radius of the circle will increase from zero to a maximum equal to the radius of the sphere (when the sphere is half full of water.) After this point the radius of the circle will decrease to zero (when there is no water in the sphere.)

These are some of the possible sketches for the shapes of the surface of the water:



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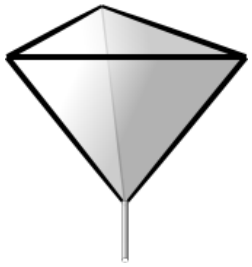
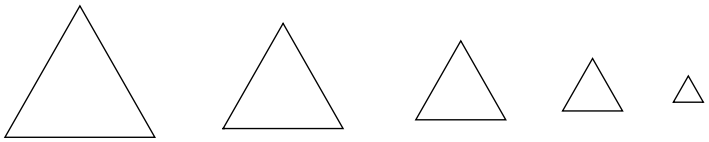


As the level of water decreases in the tetrahedron, the shape the surface of the water will always be an equilateral triangle. The side lengths of the triangle will decrease to zero (when there is no water in the tetrahedron.)

These are some of the possible sketches for the shapes of the surface of the water:



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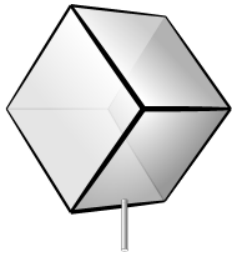


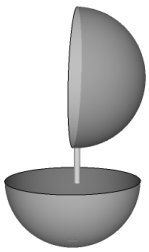
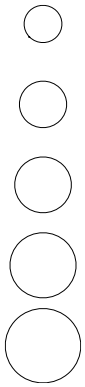
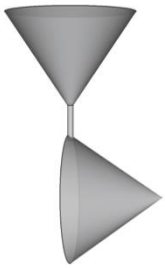
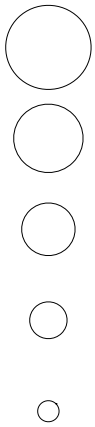
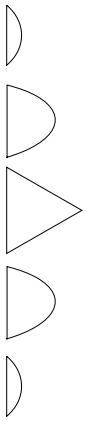
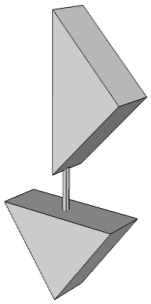
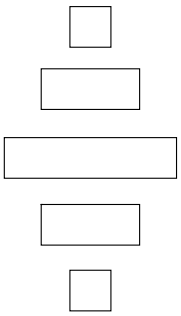
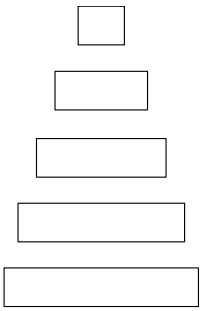
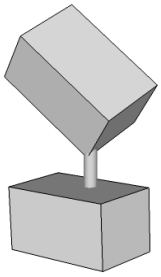
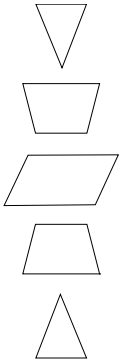
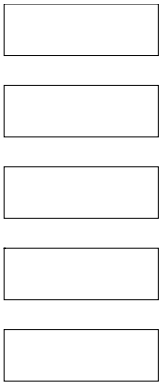
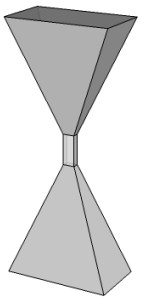
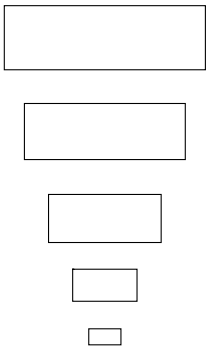
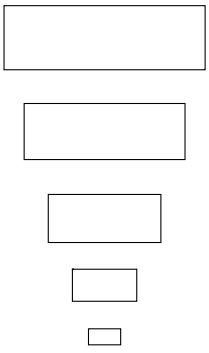
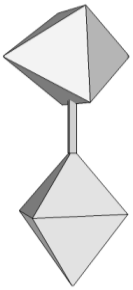
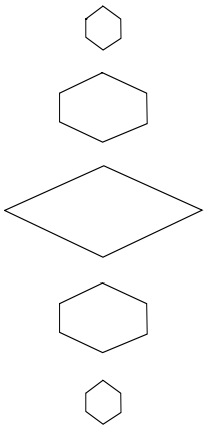
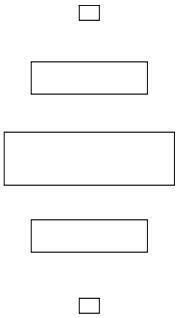
As the level of water decreases in the cube, the shape of the surface of the water will always be a rectangle. The length of the rectangle will remain equal to the length of a side of the cube, but the width will increase from zero to a maximum equal to the length of a diagonal of the cube (when the cube is half full of water.) After this point the width of the rectangle will decrease to zero (when there is no water in the cube.)

These are some of the possible sketches for the shapes of the surface of the water:

Collaborative activity

For each set of matches, the first two-dimensional diagram represents the shapes of the surface of the water as the water flows out of the top container.

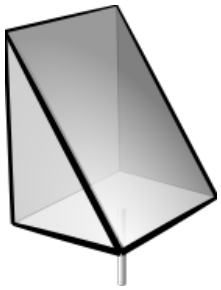
Students may have drawn in surfaces at a slightly different level to the ones shown below.

F1 	S4 	S9 	F2 	S11 	S7 
F3 	S6 	S2 	F4 	S3 	S5 
F5 	S8 or S10 	S8 or S10 	F6 	S12 	S1 

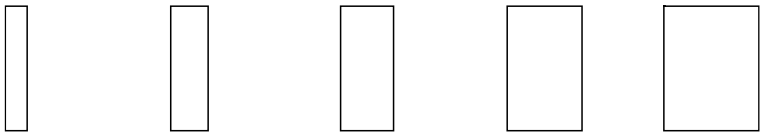
Cards S10 and S8 are interchangeable as they are exactly the same.

Assessment task: *Vessels of Water (revisited)*

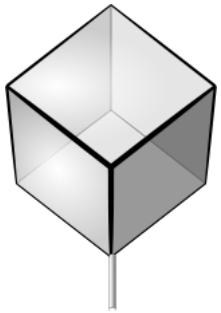
1.



As the level of water decreases in the triangular prism, the shape of the surface of the water will always be a rectangle. The length of the rectangle will remain equal to a length of a side of the 'base' of the prism, but the width will increase from zero to a maximum equal to the width of the same 'base'. These are some of the possible sketches for the shapes of the surface of the water:



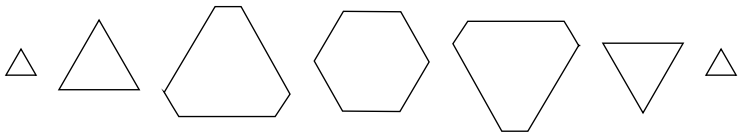
2.



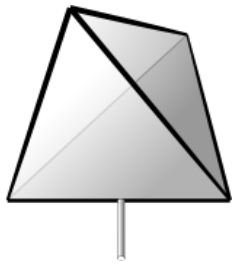
As the level of water decreases in the cube, the shape of the surface of the water will first be an equilateral triangle. The dimensions of the triangle increase to a maximum of the length of a diagonal of a cube. Then the shape of the surface of the water changes to an irregular hexagon. When the cube is half full of water the shape of the surface of the water is a regular hexagon. After this the shape of the surface of the water changes back to an irregular hexagon and then an equilateral triangle of decreasing dimensions.

Many students will not assume the cube is at 45° to the horizontal and so their shapes will not be regular.

These are some of the possible sketches for the shapes of the surface of the water:

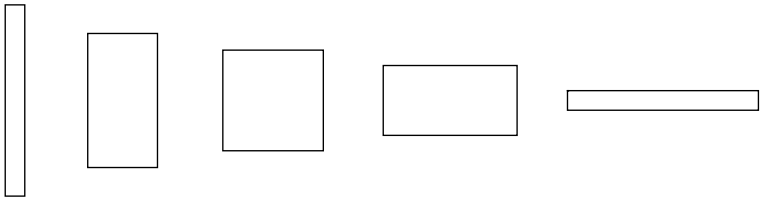


3.



As the level of water decreases in the tetrahedron, the shape of the surface of the water will be a rectangle. The length of this rectangle will initially be equal to the length of the edge of the tetrahedron and its width will be zero. The length will decrease to zero and at the same time the width will increase until it is finally equal to the length of an edge of the tetrahedron (when there is no water in the tetrahedron.) When the tetrahedron is half full of water the cross-section of the surface of the water will be a square.

These are some of the possible sketches for the shapes of the surface of the water:



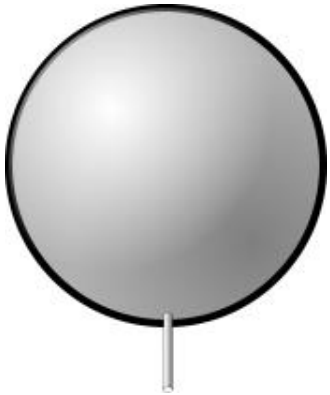
Vessels of Water

Below are representations of three-dimensional vessels. The three vessels are full of water.

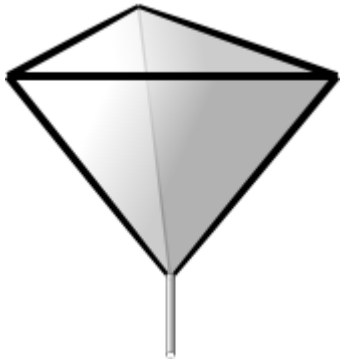
Water flows out through a pipe in the bottom of each vessel.

Sketch diagrams for each vessel to show how the shape of the surface of the water changes as the water flows out of each vessel. For each of your drawings describe the shapes formed.

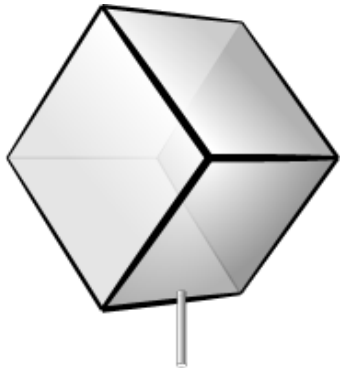
1. Sphere



2. Regular tetrahedron

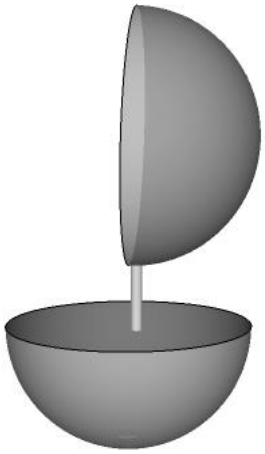


3. Cube

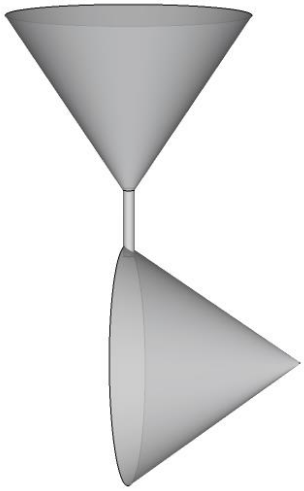


Card Set: Flowing Water

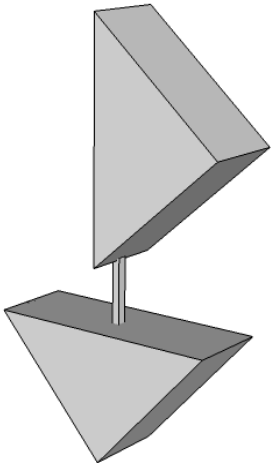
F1



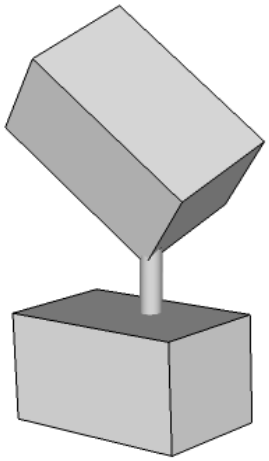
F2



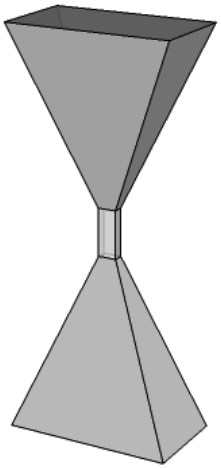
F3



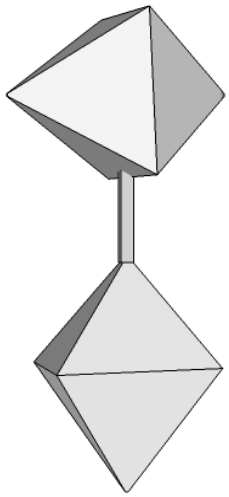
F4



F5

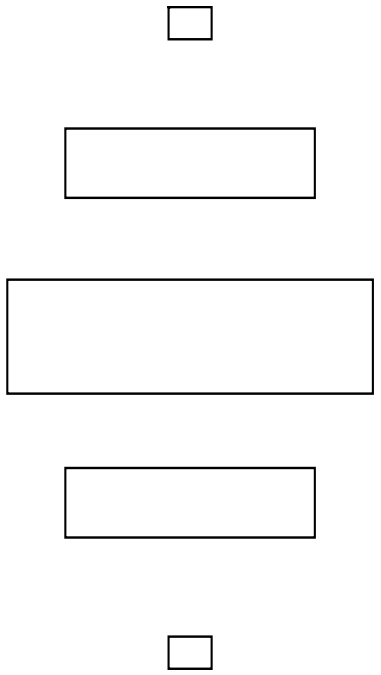


F6

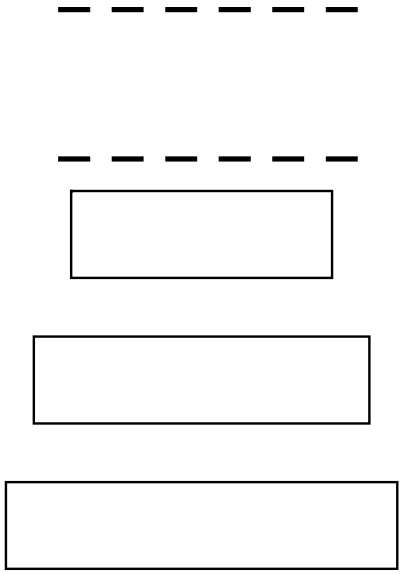


Card Set: Shape of the Surface of the Water (1)

S1

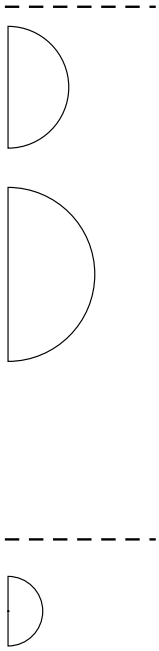


S2

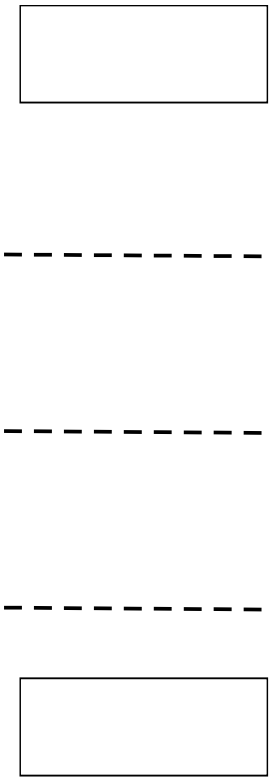


S3

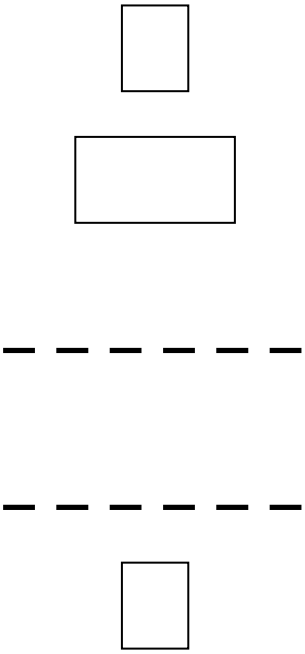
S4



S5



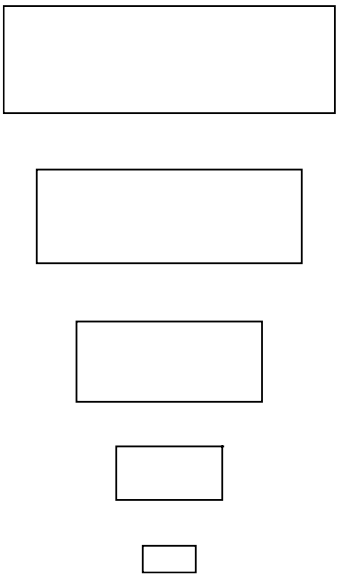
S6



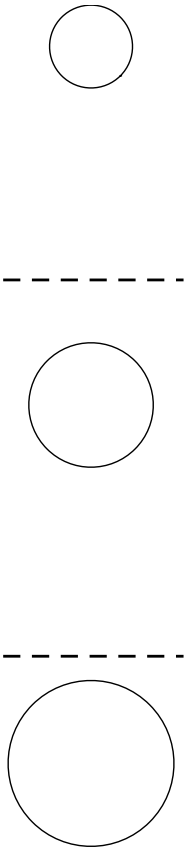
Card Set: Shape of the Surface of the Water (2)

S7

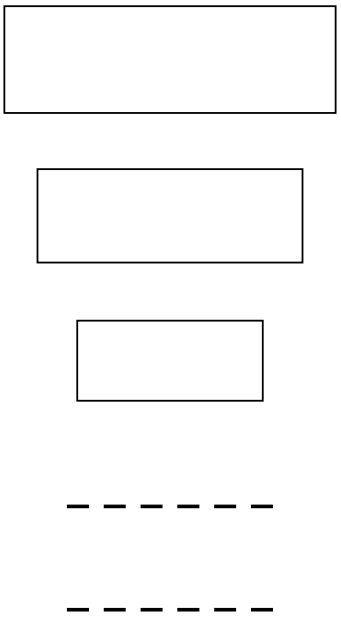
S8



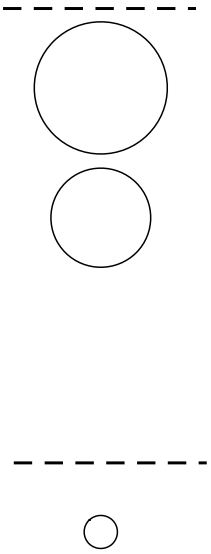
S9



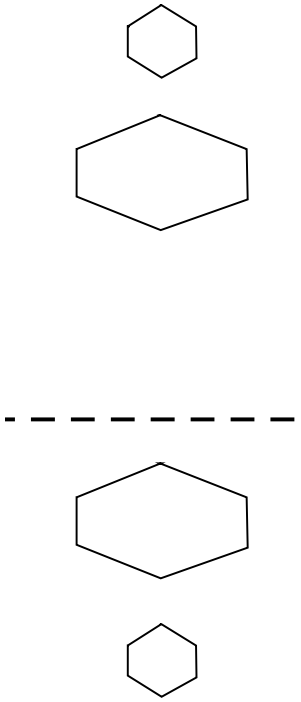
S10



S11



S12



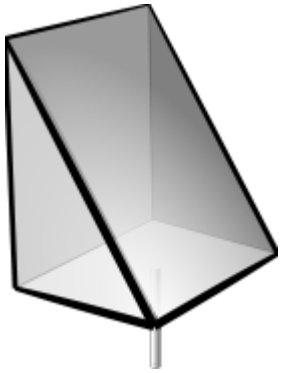
Vessels of Water (revisited)

The three vessels are full of water.

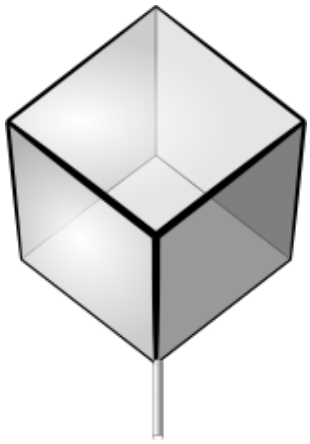
Water flows out through a pipe in the bottom of each vessel.

Sketch diagrams to show how the shape of the surface of the water changes as the water flows out of each vessel. Describe the shapes formed.

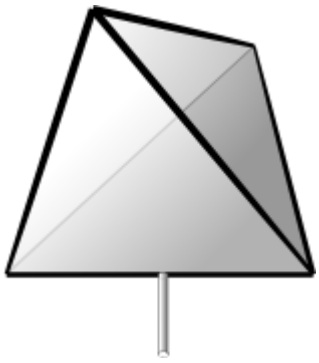
1. Triangular prism



2. Cube



3. Regular tetrahedron



Water Flowing Out of a Cylinder

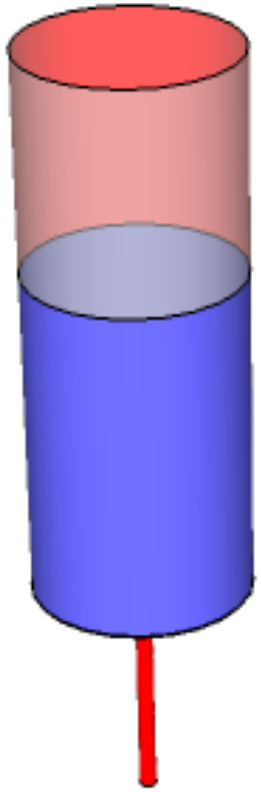
3D View:

Water Flowing Out of a Cylinder

3D View:

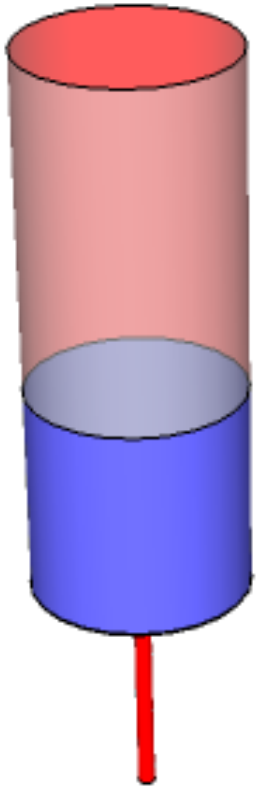
Water Flowing Out of a Cylinder

3D View:



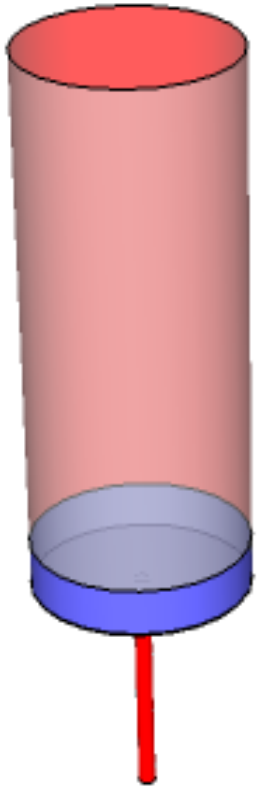
Water Flowing Out of a Cylinder

3D View:



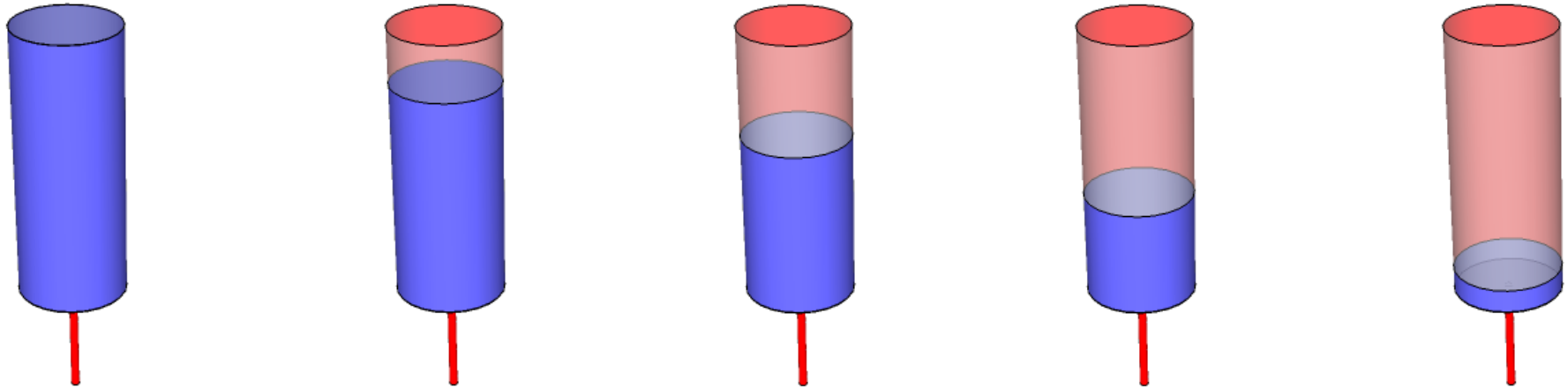
Water Flowing Out of a Cylinder

3D View:

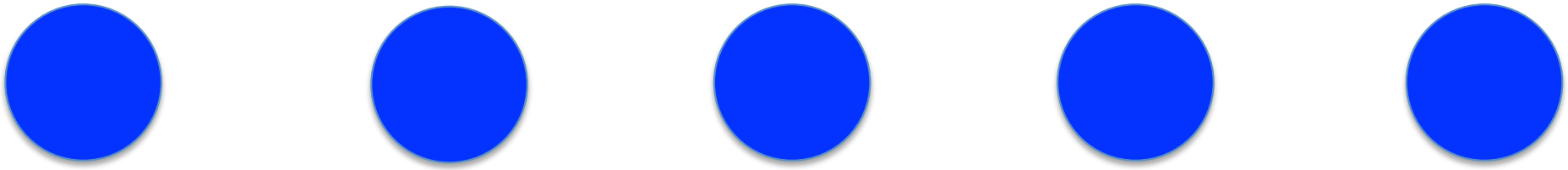


3D and 2D views

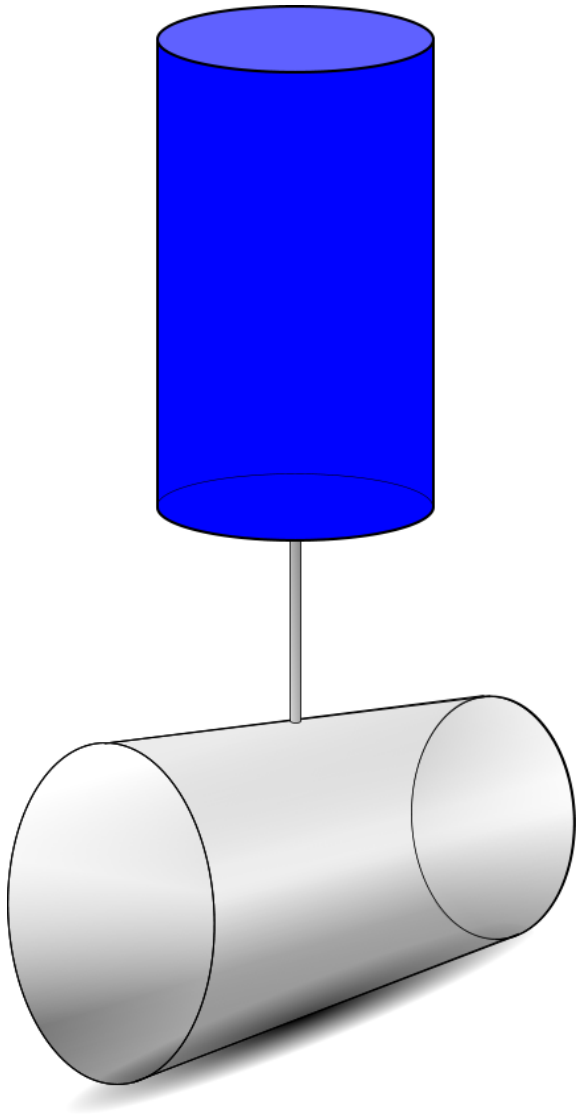
3D View:



Top view – surface of water:

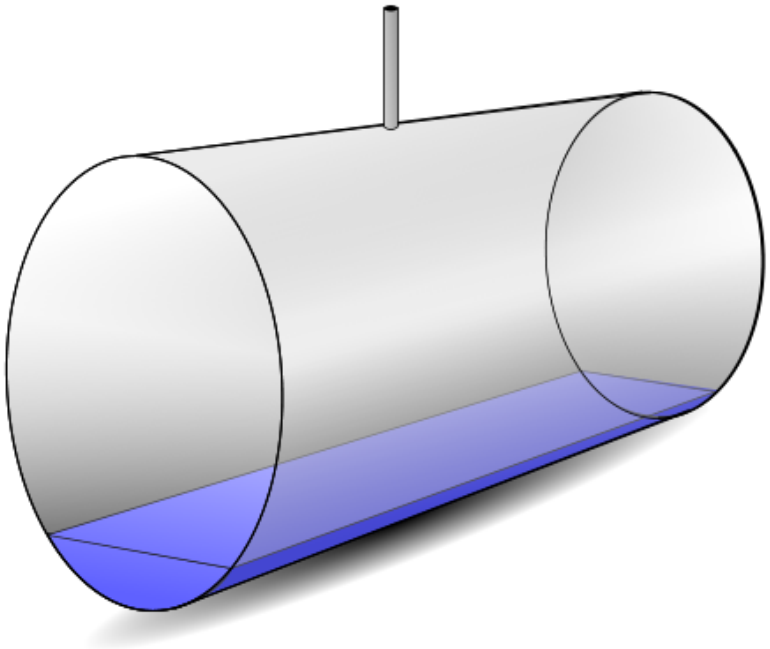


Water Flows from Top Cylinder to Bottom Cylinder



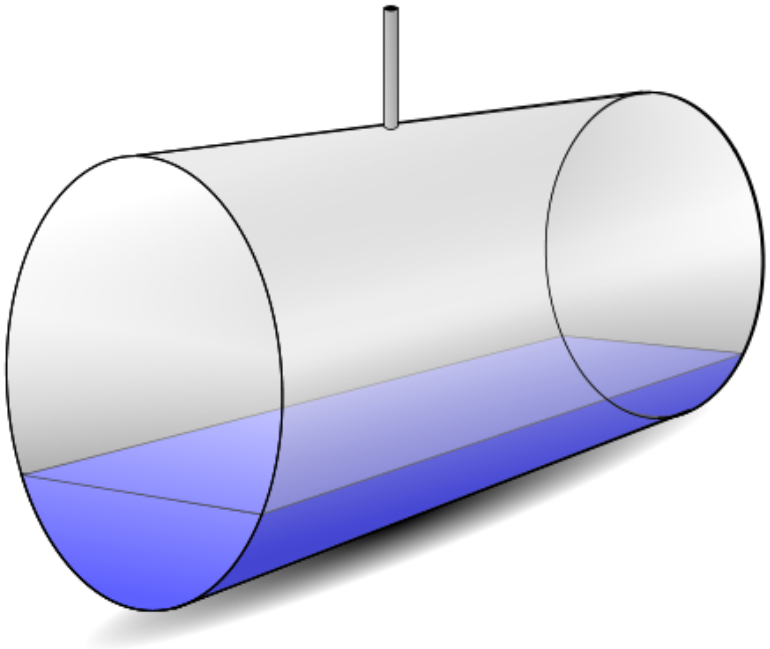
Water Flows into Bottom Cylinder

3D View:



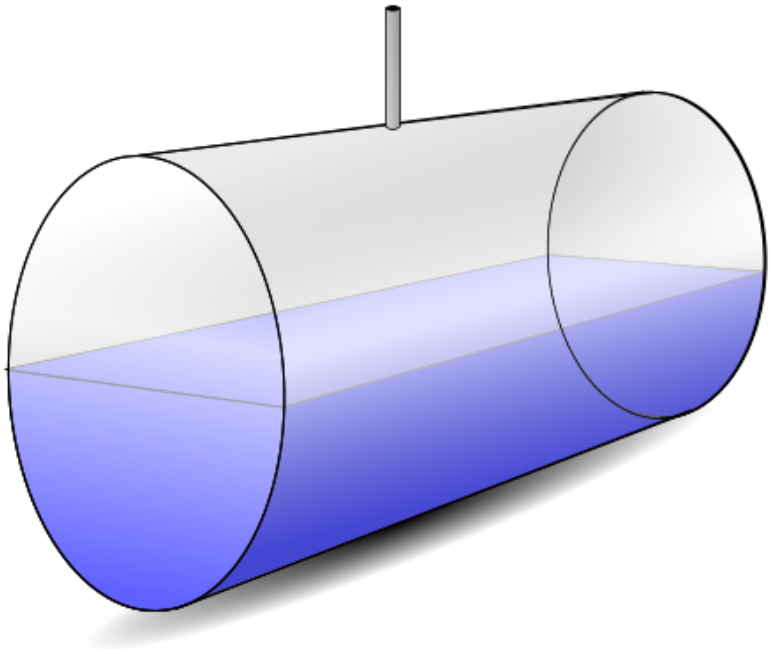
Water Flows into Bottom Cylinder

3D View:



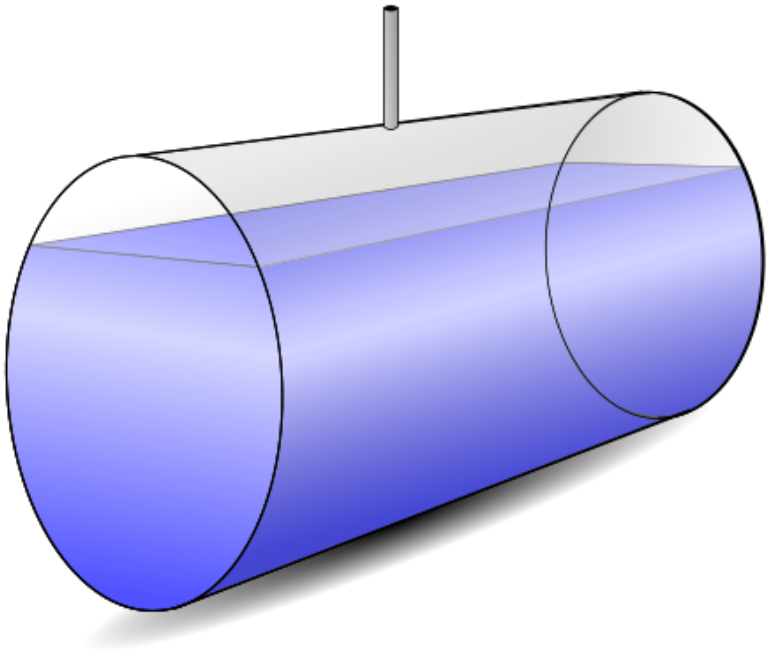
Water Flows into Bottom Cylinder

3D View:



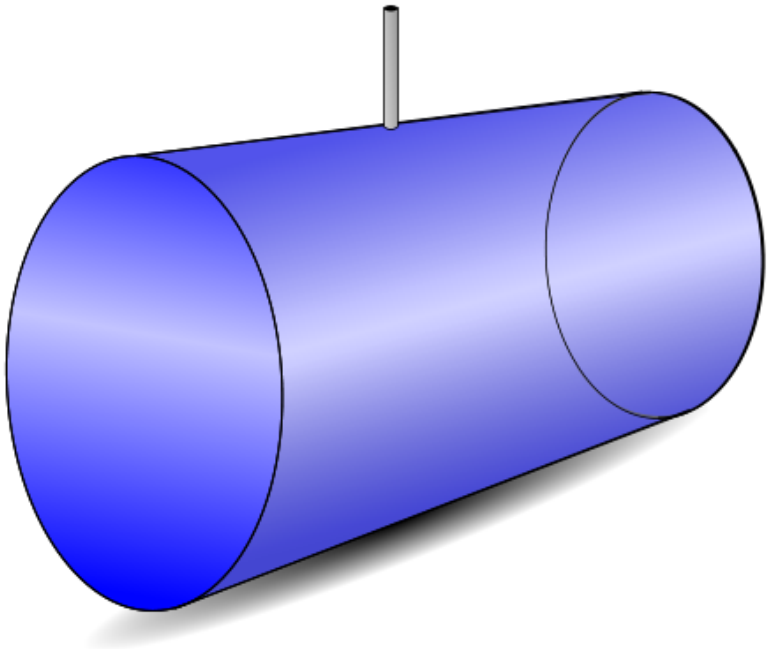
Water Flows into Bottom Cylinder

3D View:



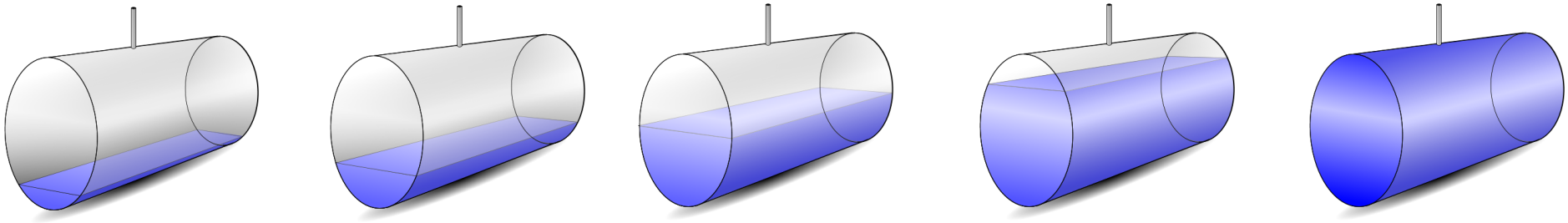
Water Flows into Bottom Cylinder

3D View:



3D and 2D Views

3D View:



Top view – surface of water:

Working Together

1. Your task is to match each set of shapes of the surface of the water with either a top or bottom container.

2. Some of the cards are missing shapes. You will need to draw these.

3. Take turns to match two cards.
   Explain to the rest of the group how you came to your decision.

4. Your partner(s) should either explain that reasoning again in his/her own words, or challenge the reasons you gave.

5. Place your cards on the poster. Next to them, record your jointly agreed justification for the match.

Everyone in your group needs to be able to agree on and explain the match of every card.

Sharing Work

1. If you are staying at your desk, be ready to explain the reasons for your group's matches.

2. If you are visiting another group, write your card matches on a piece of paper. Go to another group's desk and check to see which matches are different from your own.

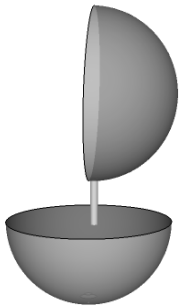
3. If there are differences, ask for an explanation. If you still don't agree, explain your own thinking.

4. When you return to your own desk, you need to consider as a group, whether to make any changes to your work.

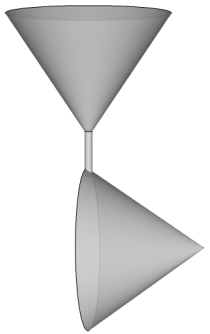
Everyone in your group needs to be able to agree on and explain the match of every card.

Card Set: Flowing Water

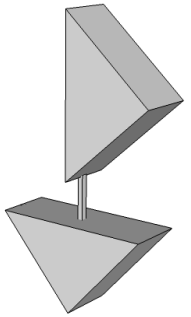
F1



F2



F3



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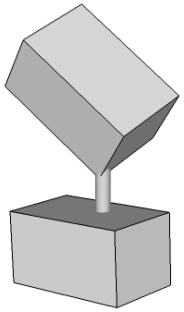
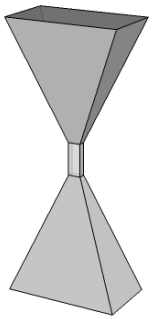
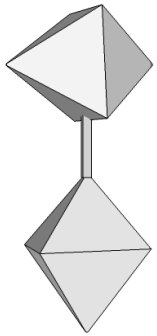


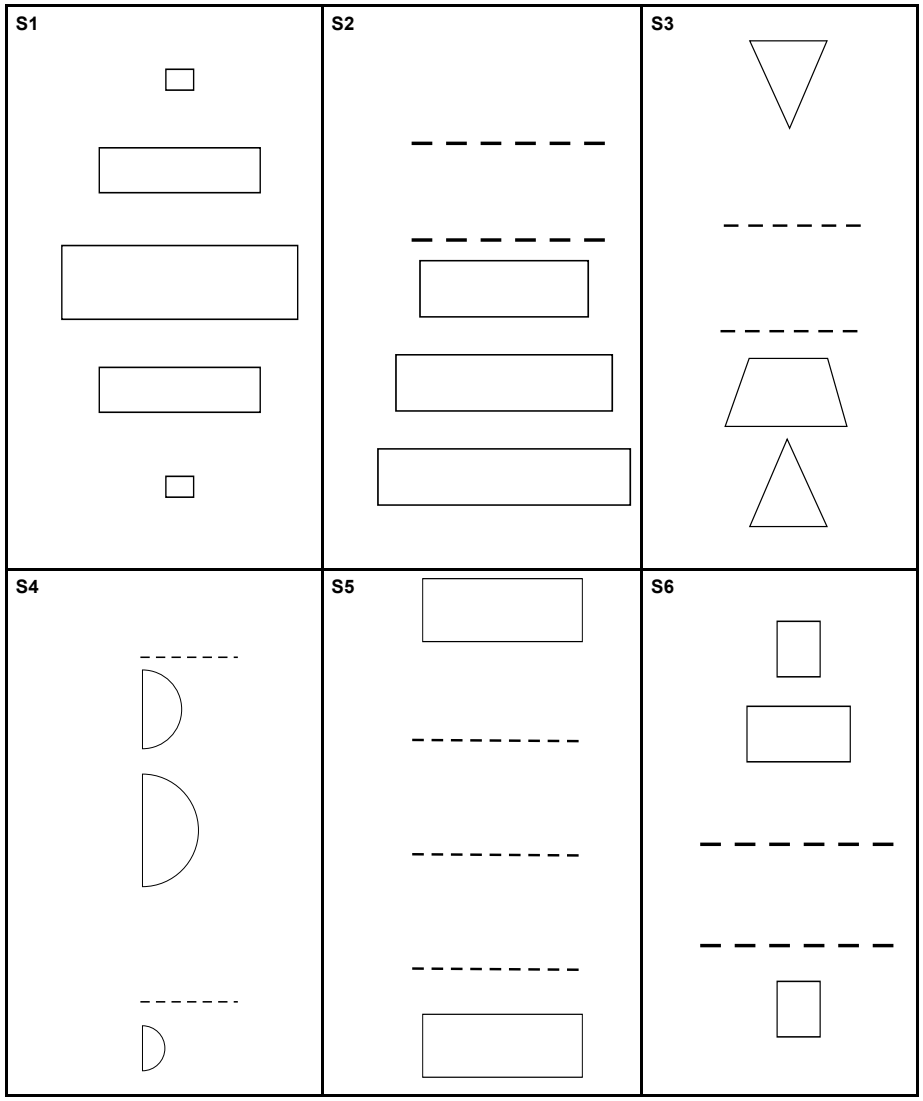
F5



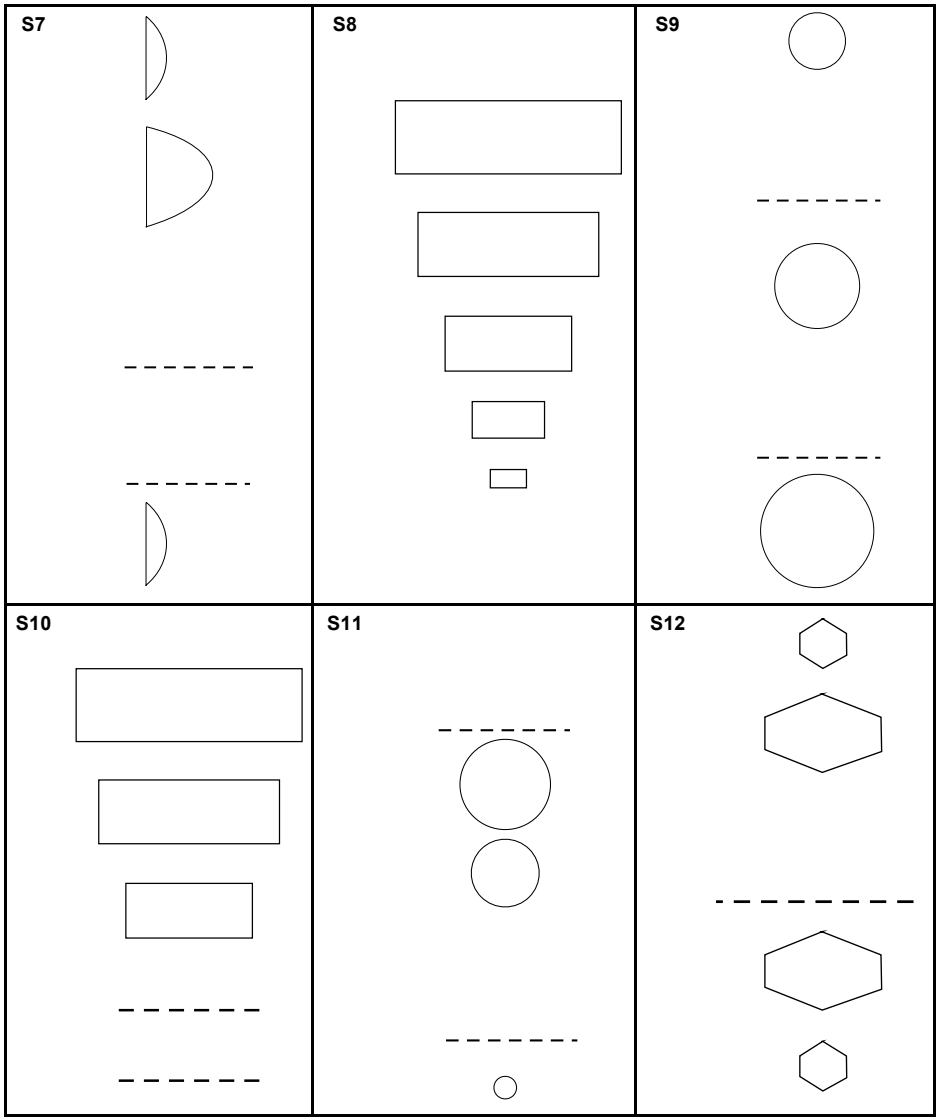
F6



Card Set: Shape of the Surface of the Water (1)



Card Set: Shape of the Surface of the Water (2)



Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org