CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Evaluating Conditions for Congruency

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Evaluating Conditions for Congruency

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Work with concepts of congruency and similarity, including identifying corresponding sides and corresponding angles within and between triangles.
- Identify and understand the significance of a counter-example.
- Prove and evaluate proofs in a geometric context.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

G-CO: Understand congruence in terms of rigid motions.
Make geometric constructions.
G-SRT: Prove theorems involving similarity.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*:

1. Make sense of problems and persevere in solving them.
3. Construct viable arguments and critique the reasoning of others.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

In this lesson, students work on the concept of congruency whilst developing their understanding of proof in a geometric context.

- Before the lesson, students complete a task designed to help you assess their current levels of understanding. You analyze their responses and write questions to help them improve their work.
- The lesson begins with a whole-class discussion about establishing conditions for congruency from triangle properties. Students work alone to decide on the truth of a conjecture about congruency conditions for triangles. Then in pairs they share ideas and produce and justify a joint response. Working in the same pairs, they analyze sample responses produced by other students. In a whole-class discussion, students develop their understanding of proof in this context.
- In a follow-up lesson students use what they have learned to improve their responses to the initial assessment task before attempting a second, similar task.

MATERIALS REQUIRED

- Each student will need a copy of *Finding Congruent Triangles* and *Finding Congruent Triangles* (*revisited*), a rule, compasses, a protractor, the cut-up *Card Set: Must the Two Triangles be Congruent?* and *Instructions*: *Must the Two Triangles be Congruent?,* a mini-whiteboard, pen, and eraser and some paper to work on.
- Each small group of students will need the *Sample Student Proofs*, some paper to work on and a glue stick.
- A board compass, protractor and rule would be useful.

TIME NEEDED

20 minutes before the lesson, a 100-minute lesson (or two 55-minute lessons), and 20 minutes in a follow-up lesson. Timings are approximate and will depend on the needs of your students.

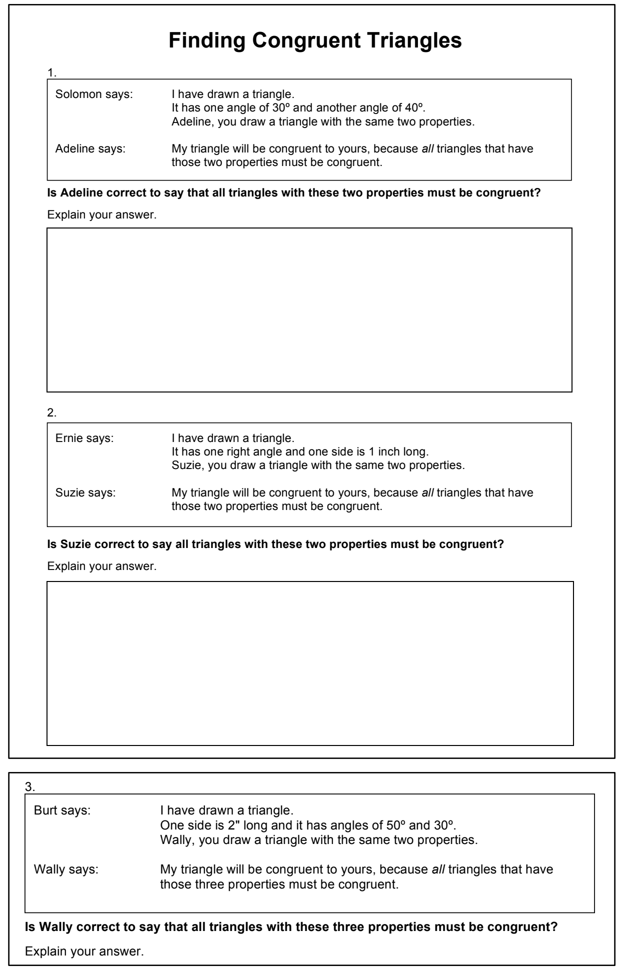
BEFORE THE LESSON

Assessment task: *Finding Congruent Triangles* (20 minutes)

Give each student the *Finding Congruent Triangles* task, a rule, compasses, and a protractor. Supply extra paper as needed.

Introduce the task briefly and help the class to understand the problem and its context:

> *You are asked to think about properties of triangles. What does 'property' mean here? [Side lengths, measures of angles and so on.]*
>
> *Suppose I have a triangle with a side 3" long and it is isosceles. Your triangle has the same properties. Must your triangle be congruent to mine? [No.] Explain why/why not.*
>
> *I would like you to work alone for 15 minutes, answering these questions.*
>
> *Think about each statement and decide whether it is true. Be sure to give good reasons for your answers. In addition to your written reasons, you might want to sketch or construct diagrams to help explain your reasoning.*


Finding Congruent Triangles

1.

Solomon says: I have drawn a triangle.
It has one angle of 30º and another angle of 40º.
Adeline, you draw a triangle with the same two properties.

Adeline says: My triangle will be congruent to yours, because *all* triangles that have those two properties must be congruent.

Is Adeline correct to say that all triangles with these two properties must be congruent?

Explain your answer.

2.

Ernie says: I have drawn a triangle.
It has one right angle and one side is 1 inch long.
Suzie, you draw a triangle with the same two properties.

Suzie says: My triangle will be congruent to yours, because *all* triangles that have those two properties must be congruent.

Is Suzie correct to say all triangles with these two properties must be congruent?

Explain your answer.

3.

Burt says: I have drawn a triangle.
One side is 2" long and it has angles of 50º and 30º.
Wally, you draw a triangle with the same two properties.

Wally says: My triangle will be congruent to yours, because *all* triangles that have those three properties must be congruent.

Is Wally correct to say that all triangles with these three properties must be congruent?

Explain your answer.

Reassure any students who do not complete the work:

> *Don't be too concerned if you did not complete all the questions. We will work on a lesson [tomorrow] that should help you improve your work.*

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and any difficulties they encountered.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare scores and distract their attention from what they might do to improve their mathematics.

Instead, help students to make progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in the trials of this unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students in the follow-up lesson.

Common issues:	**Suggested questions and prompts:**
Draws unhelpful diagrams For example: The student draws triangles that do not have the given properties.	• What properties must your triangle have? • What properties does your drawn triangle have?
Uses a restricted range of examples For example: The student's range of diagrams is restricted and a counter-example is not found.	• How can you be sure you have tried enough different triangles? • Is there a way to make these triangles so that they're not congruent? • You have drawn a triangle with the known side between the two known angles. Can you draw a triangle with a different arrangement of sides and angles?
Does not use a counter-example correctly For example: The student does not draw a diagram showing a counter-example and falsely concludes that a conjecture is true. Or: The student does not recognize a triangle he/she has drawn as a counter-example. Or: The student draws multiple counter-examples to show a general statement is false.	• You have drawn two right triangles; each has a 1" side. They are not similar to each other. What does this show you? (Q2) • In all your examples the side between the angles is the same length. Is this one of the set of properties in the question? • What would you need to do to show that the following statement is false: "All triangles with these properties are congruent"? • How many counter-examples are needed to show that a statement is false?
Uses inductive reasoning For example: The student draws several examples and concludes that if the side is contained between two angles, all the triangles with the set of properties will be congruent (Q3). Although this is true, the reasoning is mathematically weak.	• How can you be sure that there is *no* triangle for which this statement is false? • Suppose someone is not convinced by your argument. Try to find a stronger reason for supposing this statement is correct.
Provides inadequate justification For example: The student draws diagrams but does not explain them. Or: The student writes some facts and derivations, but does not join them together into a cogent proof.	• Write down in words how your work shows that the claim is true/false. • Imagine you are trying to get a student in another class to believe you are correct from what you write. How much detail will you need to write down?

SUGGESTED LESSON OUTLINE

Introduction (20 minutes)

Remind students of their work on the assessment task:

I read through your Finding Congruent Triangles scripts and have written some questions about your work.

Next lesson I'll ask you to use my questions and what you learn today to try to answer some more questions about congruency.

Give each student the cut-up *Card Set*: *Must the Two Triangles be Congruent*?, the *Instructions*: *Must the Two Triangles be Congruent?* sheet, a mini-whiteboard, pen, and eraser.
Each student will also need a rule, compasses, and a protractor.

Usually we would say sides of equal length are congruent.
Just for today, we will use 'congruent' only when we're talking about triangles, not sides or angles. That should help us to keep our ideas clear!

Explain the activity with two examples, one simple and the other more complex. There is a projectable resource to help with this discussion.

Example 1:

Ask students to look at Card 1 and project this card on the screen (Slide P-1).

1.
One side of Triangle A is the same length as one side of Triangle B.

On your mini-whiteboard, draw two triangles, Triangle A and Triangle B. Make one side of Triangle A the same length as one side of Triangle B.

Make up some side lengths and angles on your example.

Ask yourself the question: Is there a way to draw these triangles so they're not congruent?

After students have had enough time to work on the task, ask them to show each other their examples:

Take a look around. You've all drawn pairs of triangles.

Mylo, show the class your pair of triangles.
Stacey, are Mylo's triangles congruent?
Why do you think that?

Mylo's example shows us that it is possible to make Triangle A congruent to Triangle B.
Stacey's example shows that it is also possible for the triangles to be different.

Emphasize the key questions:

Must Triangle A be congruent to Triangle B?

Is it possible that they are not congruent?

Is there a way to make these triangles so they're not congruent?

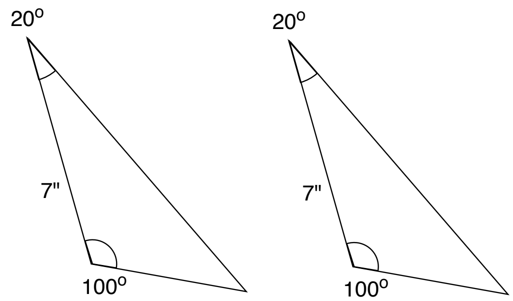
In this case, the triangles **can** be congruent but do not **have** to be congruent.

Example 2:

Ask students to consider Card 7 and project this card on the screen (Slide P-2).

Now show me a pair of triangles with these properties.

Some students will draw a pair of congruent triangles:



> 7.
>
> One side of Triangle A is the same length as one side of Triangle B
>
> and
>
> two angles in Triangle A are the same sizes as two angles in Triangle B.

Let's think again about that really important question:
Is there a way to make these triangles so they are not congruent?

After students have had enough time to work on the task, ask them to show each other their examples.

Take a look around. You've all drawn pairs of triangles.

Keely, show the class your pair of triangles.
Brent, are Keely's triangles congruent? Why do you think that?

Keely's example shows us that it is possible to make Triangle A congruent to Triangle B.
Sally's example shows that it is also possible for the triangles to be different.

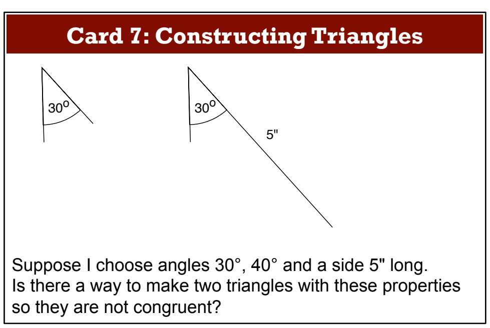
Emphasize again the key questions:

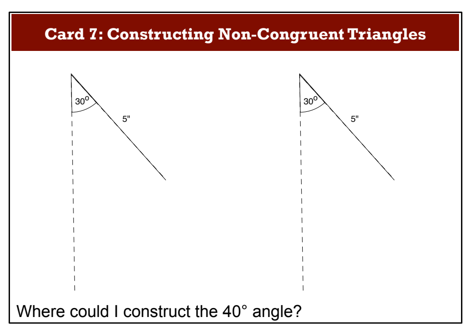
Must Triangle A be congruent to Triangle B?

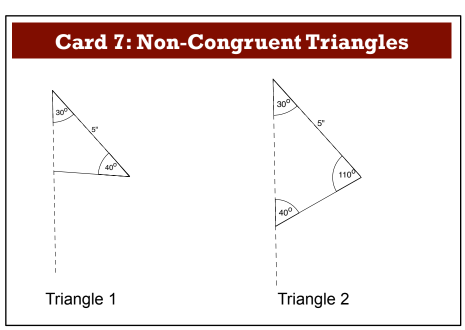
Is it possible that they are not congruent?

Is there a way to make these triangles so they're not congruent?

It may help to show students how to construct a pair of non-congruent triangles with the given properties. Slides P-3, P-4, and P-5 of the projector resources might help you do this:







Individual work: *Proving Congruency Conditions* (15 minutes)

Project the set of statements (Slide P-6) and explain the purpose of the activity:

Card Set: Must the Two Triangles be Congruent?

1. One side of Triangle A is the same length as one side of Triangle B.	2. Two sides of Triangle A are the same lengths as two sides of Triangle B.	3. Three sides of Triangle A are the same lengths as three sides of Triangle B.
4. One side of Triangle A is the same length as one side of Triangle B and one angle in Triangle A is the same size as one angle in Triangle B.	5. Two sides of Triangle A are the same lengths as two sides of Triangle B and one angle in Triangle A is the same size as one angle in Triangle B.	6. Three sides of Triangle A are the same lengths as three sides of Triangle B and one angle in Triangle A is the same size as one angle in Triangle B.
7. One side of Triangle A is the same length as one side of Triangle B and two angles in Triangle A are the same sizes as two angles in Triangle B.	**8.** Two sides of Triangle A are the same lengths as two sides of Triangle B and two angles in Triangle A are the same sizes as two angles in Triangle B.	9. Three sides of Triangle A are the same lengths as three sides of Triangle B and two angles in Triangle A are the same sizes as two angles in Triangle B.

In this activity, you are going to work alone to try and test out some more sets of properties.

You've got nine sets of properties; we've looked at two so far; Cards 1 and 7.

I would like you to choose at least two more cards, making sure one of these is Card 5.

Display Slide P-7 of the projector resource and explain how students are to work on the task:

Must the Two Triangles be Congruent?

For each card:

1. Draw examples of pairs of triangles A and B that have the properties stated in the card.
2. Decide whether the two triangles **must** be congruent. Record your decision at the bottom of the card.
3. If you decide that the triangles **do not** have to be congruent, draw examples and explain why.
4. If you decide that the triangles **must** be congruent, try to write a convincing proof.

Make sure to include Card 5 in your work, as the whole-class will discuss this statement.

Supply students with paper to work on and extra construction resources as needed.

Whilst students work, you have two tasks: to notice students' approaches to proof and to support their reasoning.

Notice students' approaches to proof

Notice how students go about drawing examples as they conjecture. Do they tend to draw triangles that are similar in shape, size, and orientation? Do they systematically vary properties and look for extreme cases?

Observe how students use their examples to reach a conclusion. Do they notice counter-examples? Do they provide evidence that they grasp the significance of a single counter-example? Do they make inductive generalizations based on a few or several cases? How typical are those cases: do they all share further, unspecified properties? Do they use transformation geometry?

What is the quality of students' written conjectures and justifications? Do they refer back to their prior knowledge of congruency conditions? Do they clarify added constraints? Do they explain why their examples are relevant? Do students write down enough of their reasoning to make the solution transparent for the reader?

You can use the information you gather to support students' reasoning and also to focus the whole-class discussion at the end of the lesson.

Support students' reasoning

Try to support students in their proving activities, rather than steering them towards any particular method of proof. Prompt students to consider a wide range of examples, including variation in the relative positions of sides and angles:

> *Does it make any difference where the sides go and where the angle goes? How do you know?*
> *Have you drawn a wide range of possible pairs of triangles? How did you choose your examples?*

Encourage students to write down their explanations, giving clear references to diagrams, using labeling and including lots of small details.

> *You've drawn some examples of triangles with that set of properties. What conclusion did you reach? Can you explain that in writing?*
>
> *Imagine you are writing this explanation for a teacher in another class. How much do you need to write to make your reasoning clear for a reader?*
>
> *Have you considered adding a diagram? Explain to your reader how the diagram will help.*

Help students to clarify the meaning of the questions and to use clear language. Encourage them to clarify the logic of the conditions on which they work.

You may also want to use questions like those in the *Common issues* table to help students address their errors and misconceptions.

Small-group work: producing a joint response (20 minutes)

When everyone has produced some work on at least two sets of properties, organize students into pairs or small groups of three.

Give each group two or three sheets of paper and a glue stick.

Ask students to take turns to explain their work to other members of the group before working together to reach an agreed conclusion:

> *Take turns. When it is your turn, select a card you have worked on. Glue it to the middle of a blank sheet of paper.*
>
> *Explain your work to the others. Explain your conclusion and how you reached that conclusion. Make sure everyone in your group understands your diagrams.*
>
> *If anyone else in the group has worked on that same card, s/he should then explain their reasoning and conclusion.*
>
> *Work together to reach an agreed conclusion. On your sheet write an explanation together that is better than your individual explanations.*
>
> *Make sure you discuss Card 5.*

Slide P-8 of the projector resource, *Working Together*, summarizes these instructions.

While students are working in groups, observe, listen, and support students as before.

When talking, students may point and use terms such as 'this angle' and 'that one' rather than producing labels. It may be helpful to ask questions to get them to say the name of the triangle and to use the labels for angles and sides. If students do this when they talk, it may help them to produce clearer written work.

Encourage students to share their thinking in their groups and support those who are listening. Encourage joint 'sense-making': if one student provides some reasoning, ask the other to state it or write it in his/her own words. Then ask the first student if the second's representation was correct.

Extending the lesson over two days

If you are taking two days to complete the lesson unit then you may want to end the first lesson here. At the start of the second day, allow time for students to familiarize themselves with their joint response to Card 5. Once they have reminded themselves of their explanations and agreed conclusion they can then go on to analyze the *Sample Student Proofs*.

Small-group work: analyzing congruency proofs (20 minutes)

Check that students have had enough time to work together on at least two sets of properties, including Card 5. Then give each small group of students a copy of the two *Sample Student Proofs*. This work shows two attempts to tackle Card 5.

> *Here is some work on this set of properties produced by students in another class.*
>
> *Work in your pairs on one piece of work at a time. Read the work carefully and try to figure out what the explanation is about.*
>
> *Answer the questions about the student's reasoning that are written on the sheet. Try to help each other understand and then critique what that student was doing.*

5.
Two sides of Triangle A are the same lengths as two sides of Triangle B
and
one angle in Triangle A is the same size as one angle in Triangle B.

During this small-group work, observe and support students as before.

Notice students' analytical activities

Do they notice the assumptions that Jorge has made? Do they identify that when a further constraint is added, the angle is between the two sides (SAS), all the triangles will be congruent?

Do students understand what Kieran's diagram shows? Do they recognize his work as a counter-example to the conjecture that all the triangles with these properties are congruent? Do they identify the relevance and power of a single counter-example?

What connections do they make between the two proofs?

Support student proving activities

Ask students to explain the reasoning in the *Sample Student Proofs* and in particular, the links between words and diagrams:

> *Why do you think Kieran drew two circles? What connection do circles have with the set of triangle properties he worked with?*
>
> *What do you know from other math lessons about translations? Constructions? How does that information help you understand and critique Jorge's response?*

Ask students questions to guide their evaluation of the *Sample Student Proofs:*

> *Do you agree with [Jorge's/Kieran's] conclusion? Why/Why not?*
> *Can they both be correct? Why/Why not?*
>
> *You have done a lot of thinking together to understand this proof. How much of that thinking was written in the Sample Student Proof? How much extra did you have to do?*

Whole-class discussion (25 minutes)

Tell students that you are going to finish the lesson with a discussion, beginning with Card 5. They will need to make use of their mini-whiteboard, pen, and eraser during this discussion.

Project Slide P-9 of the projector resource. Ask students to say whether they thought the two triangles with properties specified on Card 5 must be congruent. If there is disagreement, point this out to students and tell them that you will work to come to a joint conclusion.

Ask students to first look at Jorge's proof.

Does Jorge think that the statement on Card 5 is true or false? [True.]

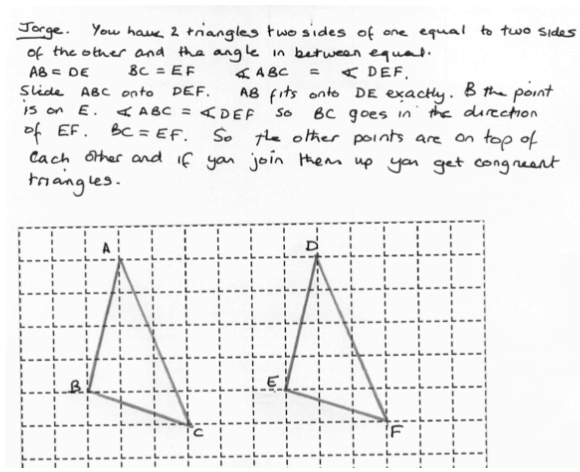
Do you agree with Jorge? Why/Why not?

What extra property does Jorge bring in?

How does this affect the statement on Card 5? [Jorge only considers some of the triangles that have these properties, not all of them. He only considers triangles where SAS are equal; he ignores SSA].

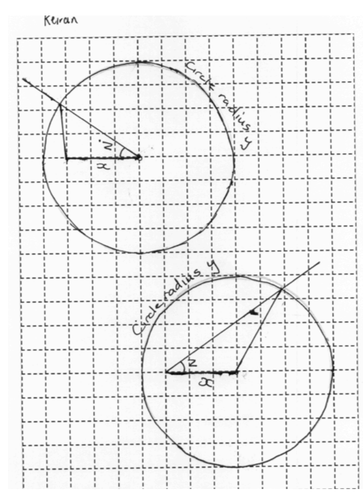
It may be helpful to use Slides P-10 and P-11, showing Jorge's and Kieran's proofs during this discussion. An analysis of the two proofs and the connections between them is given below:

Jorge adds an unacknowledged constraint to the given conditions. Jorge's solution is a proof of the congruency condition SAS. Without further information we cannot know whether Jorge has realized that knowing two sides and one angle does not, in general, specify congruent triangles.



(Jorge's proof is intended to approximate Proposition 4 in Euclid's *Elements* (Book 1)). The technique of the proof is to show that by 'applying' or imposing one triangle on another, the parts of one triangle can be shown to correspond to the parts in another. This is not an uncontroversial method, but is intuitively acceptable for students and perhaps could be seen as a use of the rigid transformation; translation.)

Kieran does little to communicate his reasoning to the reader. This is a major weakness with his proof; he makes no effort to explain his reasoning.



Kieran provides a clever counter-example by producing two triangles that have the required properties but are not congruent. The first side, x, is fixed and a circle is drawn with radius $y > x$. This is the locus of the point for the vertex of the known side y not affixed to x. The fixed angle, z, can either be placed between the two known sides or just attached to one of them.

What assumption does Jorge add?

Sketch a circle diagram like Kieran's, showing a triangle that has the properties from Card 5 plus Jorge's extra assumption.

Once students have reached a reasoned agreement about Card 5, work on strategic thinking. Project Slide P-6 of the projector resource again:

Card Set: Must the Two Triangles be Congruent?

1. One side of Triangle A is the same length as one side of Triangle B.	2. Two sides of Triangle A are the same lengths as two sides of Triangle B.	3. Three sides of Triangle A are the same lengths as three sides of Triangle B.
4. One side of Triangle A is the same length as one side of Triangle B and one angle in Triangle A is the same size as one angle in Triangle B.	5. Two sides of Triangle A are the same lengths as two sides of Triangle B and one angle in Triangle A is the same size as one angle in Triangle B.	6. Three sides of Triangle A are the same lengths as three sides of Triangle B. and one angle in Triangle A is the same size as one angle in Triangle B.
7. One side of Triangle A is the same length as one side of Triangle B and two angles in Triangle A are the same sizes as two angles in Triangle B.	8. Two sides of Triangle A are the same lengths as two sides of Triangle B and two angles in Triangle A are the same sizes as two angles in Triangle B.	9. Three sides of Triangle A are the same lengths as three sides of Triangle B. and two angles in Triangle A are the same sizes as two angles in Triangle B.

We looked together at Cards 1 and 7 at the beginning of the lesson. We've now also looked together at Card 5.

Consider all the cards together.
Which cards do you think will be simplest to decide on? Why?
[It may be easier to work with fewer properties. Once a card with a set of properties has been shown to determine congruency (e.g. three sides), all other cards using that set of properties will also ensure congruency. It's easier to disprove than prove, as you only need a single counter example.]

Ask students to explain their reasoning when they make a claim. Check whether those listening understand these explanations. If someone agrees, ask them to explain in their own words. If a student does not agree, they can explain why.

Now choose one of the cards not yet discussed in detail. Ask students to work on this on their mini-whiteboards. Then ask students to share their reasoning.

Show me what you have written so far about Card 3.
Ting, you think triangles with these properties must be congruent. Tell us why.
Alan, do you agree? Explain your answer.

It is more important that students explore a few cards in detail than that the whole table is finished.

To end the lesson, ask students to explain how to show that a general statement is false. Try to make sure they understand the notion of a counter-example, the effect of a counter-example on the truth of a generalization, and that only a single counter-example is required.

Follow-up lesson: *Finding Congruent Triangles (revisited)* (20 minutes)

Give students their work on the first assessment task, together with your questions on their work. If you have chosen not to write questions on individual scripts, write your list of questions on the board.

Give each student a copy of the second assessment task: *Finding Congruent Triangles (revisited)*. Supply students with extra paper and construction resources as needed.

Look at your original response, read my questions carefully and think about what you have learned this lesson. Spend a few minutes answering my questions and revise your response.

Now, using what you have learned, try to answer the questions on the new task, Finding Congruent Triangles (revisited).

When you think your work is complete, explain what you think a counter-example is. Explain in words, using examples and diagrams if you wish.

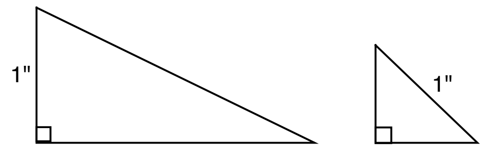
Some teachers give this as a homework task.

SOLUTIONS

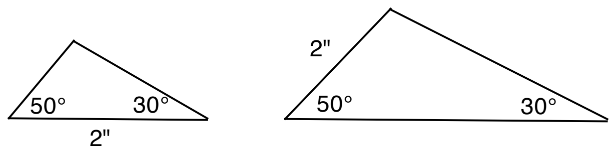
Assessment task: *Finding Congruent Triangles*

1. Adeline is incorrect: it is not true that all triangles with angles of 30º and 40º must be congruent. The student could show this with a pair of triangles, showing two similar triangles of unequal size. Students might notice that the triangles are similar but this is not part of the question.

2. Suzie is incorrect: it is not true that all right triangles with one side 1" long must be congruent. The student could show this by drawing a simple counter example:



3. Wally is incorrect: it is not true that all triangles with a side 2" long and angles of 50º and 30º are congruent. A single counter-example is again sufficient:



   Students may have unintentionally imposed an additional constraint, that the fixed-length side is contained between the two angles. If the triangles are 'AAS' they will always be similar, but not necessarily congruent. The student could explain this in terms of the fixed angle sum of triangles (so two angles entails similarity) and then exploring rigid transformations of the images.

   If the triangles are 'ASA' then they must always be congruent. This could be argued by considering rigid transformations (reflections, translations, rotations) of a diagram and showing that all images are congruent. Just drawing a few diagrams without the transformational reasoning would be an alternative weaker method of justification.

Assessment task: *Finding Congruent Triangles (revisited)*

These solutions are analogous to those above, so we will not repeat all the reasoning.

1. Ben is incorrect: it is not true that all triangles with angles of 70º and 50º must be congruent.

2. Sanjay is incorrect: it is not true that all triangles with an angle of 60° and a side length of 2" must be congruent.

3. Max is incorrect: it is not true that all triangles with a side 1" long and angles of 90º and 20º are congruent.

Main lesson activity: *Must the Two Triangles be Congruent?*

The table on the next page shows that the only conditions that ensure congruency are given in the right hand column, where the three sides of triangle A are the same lengths as the three sides of triangle B. In all other cases, it is not necessary for the triangles to be congruent and a single counter-example will prove this each time. Below the table we offer some notes on particular cards, referring to the numbers in the cards.

1. One side of triangle A is the same length as one side of triangle B. *Not necessarily congruent.*	2. Two sides of triangle A are the same lengths as two sides of triangle B. *Not necessarily congruent.*	3. All three sides of triangle A are the same lengths as all three sides of triangle B. *Must be congruent.*
4. One side of triangle A is the same length as one side of triangle B and one angle in triangle A is the same size as one angle in triangle B. *Not necessarily congruent.*	5. Two sides of triangle A are the same lengths as two sides of triangle B and one angle in triangle A is the same size as one angle in triangle B. *Not necessarily congruent.*	6. All three sides of triangle A are the same lengths as all three sides of triangle B and one angle in triangle A is the same size as one angle in triangle B. *Must be congruent.*
7. One side of triangle A is the same length as one side of triangle B and two angles in triangle A are the same sizes as two angles in triangle B. *Not necessarily congruent. The triangles are necessarily similar.*	8. Two sides of triangle A are the same lengths as two sides of triangle B and two angles in triangle A are the same sizes as two angles in triangle B. *Not necessarily congruent. The triangles are necessarily similar.*	9. All three sides of triangle A are the same lengths as all three sides of triangle B and two angles in triangle A are the same sizes as two angles in triangle B. *Must be congruent.*

Card 3

Students may try to show this by drawing several pairs of examples. Reasoning inductively, they might conclude that every pair of triangles with these properties will be congruent. Inductive reasoning is not mathematically compelling, but there is research evidence that suggests many students gain conviction this way. A fairly compelling proof accessible to students in High School would be to start to construct a generic example – a typical triangle that does not have extra special properties such as angles of equal size – with two fixed side lengths. The student can then consider how many different triangles it is possible to make given a third, fixed length.

Card 5

This is the example considered by Jorge and Kieran and discussed in detail in the final whole-class discussion.

Strictly speaking, a single counter-example will establish that the two triangles need not be congruent. However, students may find this counter-example quite difficult to find. The relative positions of angle and sides matter here: if the angle is contained by the two sides, the triangles must be congruent. Counter-examples occur when the angle is not contained between the two sides. So adding an extra property – the containment of the angle between the two sides – produces the familiar 'SAS' condition.

Card 6

This follows from Card 3. This follows from the fact that three side lengths determine a triangle. Adding a further property, an angle, will not change that.

Card 7

This example is considered in the first whole-class discussion.

These triangles must be similar because the angles are all equal. A counter-example to congruency can be produced by scaling, to change the position of the side of fixed length relative to the angles. Containing the known side between the two known angles – adding the extra property or constraint – produces the familiar 'ASA' congruency condition.

Card 8

This is perhaps the most difficult example. First notice that knowing two angles, we know three, so any counter-example will involve similar triangles.

A suitable counter-example is any single pair of similar triangles in which two sides are the same lengths in each triangle, but they are non-equivalent sides 'rotated' around the triangle, for example:

We do not expect students to generalize this, but it is interesting that in general, the scale factor of enlargement must be less than the "golden ratio":

These side lengths only specify a triangle if:

$a+\frac{a}{k}>ak$ (Assuming a > 0 and k > 0).

$$\Rightarrow k<\frac{1+\sqrt{5}}{2}$$

Card 9

This follows from the fact that three side lengths determine a triangle. Adding a further property will not change that.

Finding Congruent Triangles

1.

Solomon says:	I have drawn a triangle. It has one angle of 30º and another angle of 40º. Adeline, you draw a triangle with the same two properties.
Adeline says:	My triangle will be congruent to yours, because *all* triangles that have those two properties must be congruent.

Is Adeline correct to say that all triangles with these two properties must be congruent?

Explain your answer.

2.

Ernie says:	I have drawn a triangle. It has one right angle and one side is 1 inch long. Suzie, you draw a triangle with the same two properties.
Suzie says:	My triangle will be congruent to yours, because *all* triangles that have those two properties must be congruent.

Is Suzie correct to say all triangles with these two properties must be congruent?

Explain your answer.

3.

Burt says:	I have drawn a triangle. One side is 2" long and it has angles of 50º and 30º. Wally, you draw a triangle with the same two properties.
Wally says:	My triangle will be congruent to yours, because *all* triangles that have those three properties must be congruent.

Is Wally correct to say that all triangles with these three properties must be congruent?

Explain your answer.

Card Set: Must the Two Triangles be Congruent?

1. One side of Triangle A is the same length as one side of Triangle B.	2. Two sides of Triangle A are the same lengths as two sides of Triangle B.	3. Three sides of Triangle A are the same lengths as three sides of Triangle B.
4. One side of Triangle A is the same length as one side of Triangle B and one angle in Triangle A is the same size as one angle in Triangle B.	5. Two sides of Triangle A are the same lengths as two sides of Triangle B and one angle in Triangle A is the same size as one angle in Triangle B.	6. Three sides of Triangle A are the same lengths as three sides of Triangle B and one angle in Triangle A is the same size as one angle in Triangle B.
7. One side of Triangle A is the same length as one side of Triangle B and two angles in Triangle A are the same sizes as two angles in Triangle B.	8. Two sides of Triangle A are the same lengths as two sides of Triangle B and two angles in Triangle A are the same sizes as two angles in Triangle B.	9. Three sides of Triangle A are the same lengths as three sides of Triangle B and two angles in Triangle A are the same sizes as two angles in Triangle B.

Instructions: Must the Two Triangles be Congruent?

For each card:

- Draw examples of pairs of triangles A and B that have the properties stated on the card.
- Decide whether the two triangles **must** be congruent and record your decision.
- If you decide that the triangles **do not** have to be congruent, draw examples and explain why.
- If you decide that the triangles **must** be congruent, try to write a convincing proof.

Sample Student Proofs

Kieran and Jorge both investigated Card 5:

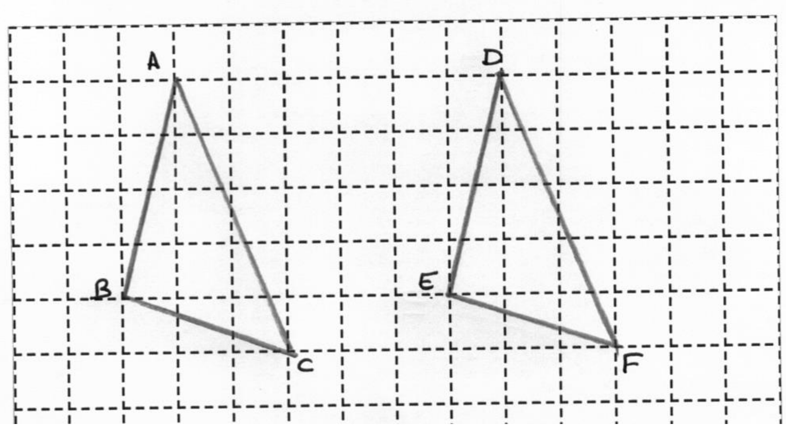
> Two sides of Triangle A are the same lengths as two sides of Triangle B
> and
> one angle in Triangle A is the same size as one angle in Triangle B.

Jorge says the triangles must be congruent.

Jorge. You have 2 triangles two sides of one equal to two sides of the other and the angle in between equal.

$AB = DE$ $BC = EF$ $\sphericalangle ABC = \sphericalangle DEF$.

Slide ABC onto DEF. AB fits onto DE exactly. B the point is on E. $\sphericalangle ABC = \sphericalangle DEF$ so BC goes in the direction of EF. $BC = EF$. So the other points are on top of each other and if you join them up you get congruent triangles.

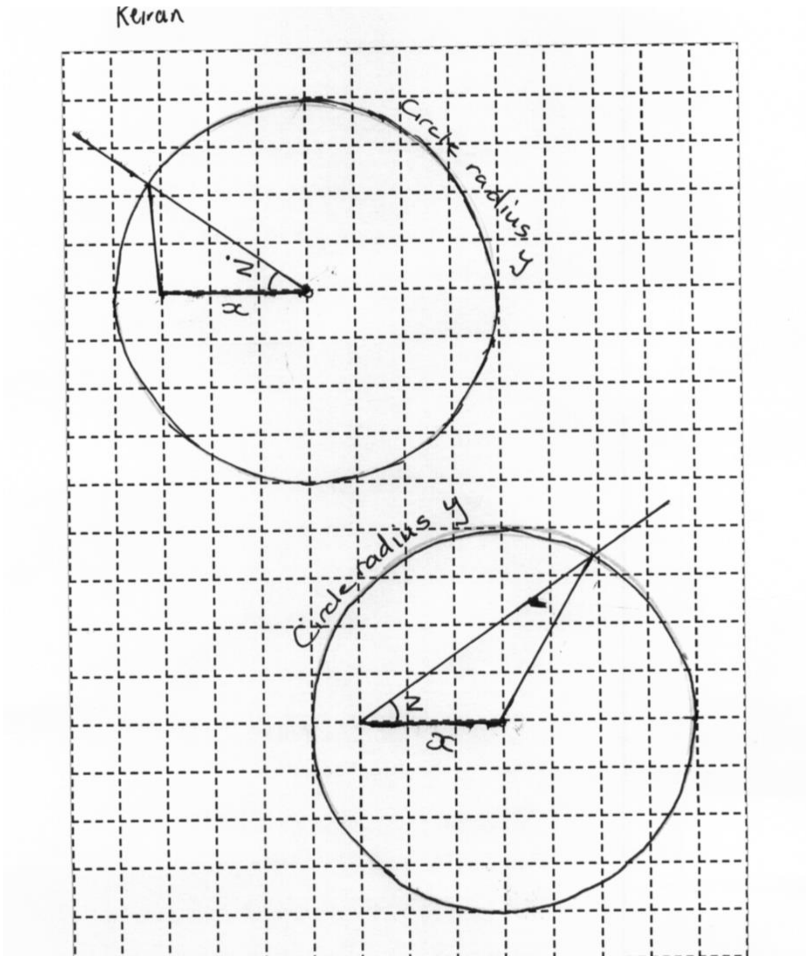


Do you agree with Jorge?

Explain Jorge's reasoning.

Explain how Jorge could improve his proof.

Kieran says the triangles need not be congruent.



Do you agree with Kieran?

Explain what Kieran's diagrams show.

Explain how Kieran could improve his reasoning.

Finding Congruent Triangles (revisited)

1.

Amy says:	I have drawn a triangle. It has one angle of 70º and another angle of 50º. Ben, you draw a triangle with the same two properties.
Ben says:	My triangle will be congruent to yours, because *all* triangles that have those two properties must be congruent.

Is Ben correct to say that all triangles with these two properties must be congruent?

Explain your answer.

2.

Sara says:	I have drawn a triangle. It has one angle of 60º and one side 2 inches long. Sanjay, you draw a triangle with the same two properties.
Sanjay says:	My triangle will be congruent to yours, because *all* triangles that have those two properties must be congruent.

Is Sanjay correct to say all triangles with these properties must be congruent?

Explain your answer.

3.

Burt says:	I have drawn a triangle. One side is 1" long, and it has angles of 90º and 20º. Max, you draw a triangle with the same two properties.
Max says:	My triangle will be congruent to yours, because *all* triangles that have those three properties must be congruent.

Is Max correct to say that all triangles with these properties must be congruent?

Explain your answer.

Card 1

1.

One side of Triangle A is the same length as one side of Triangle B.

Card 7

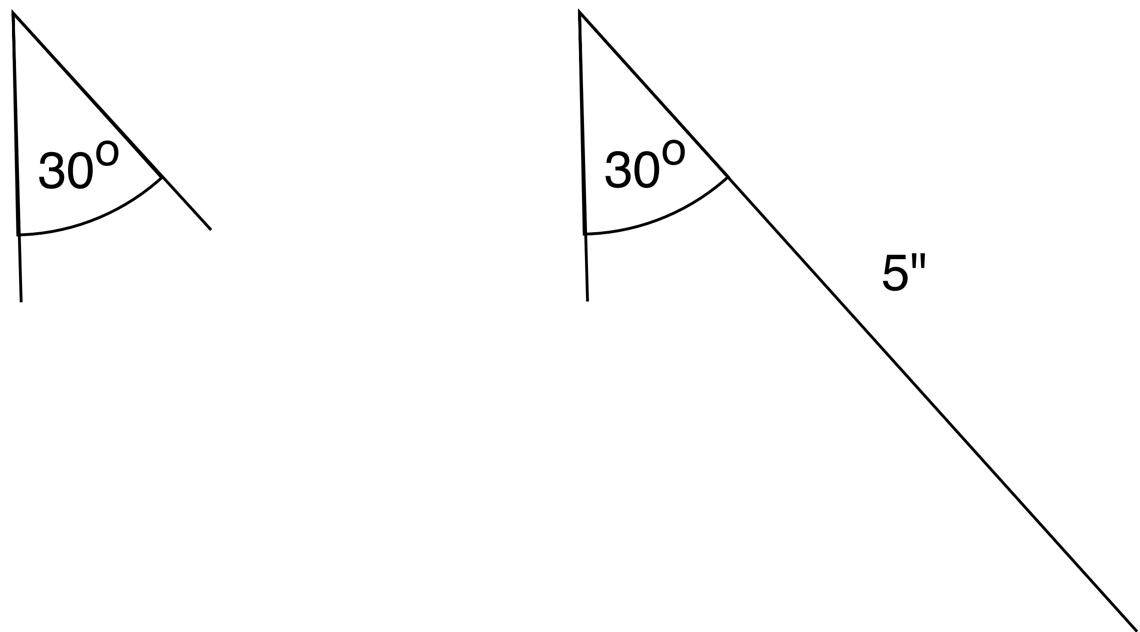
7.

One side of Triangle A is the same length as one side of Triangle B

and

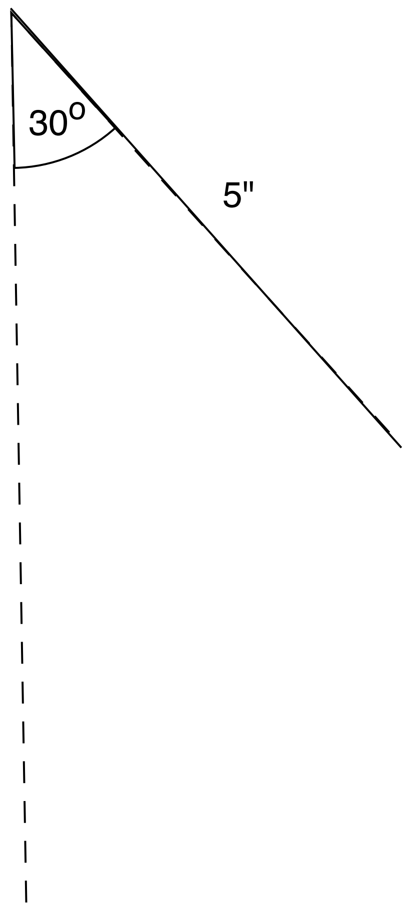
two angles in Triangle A are the same sizes as two angles in Triangle B.

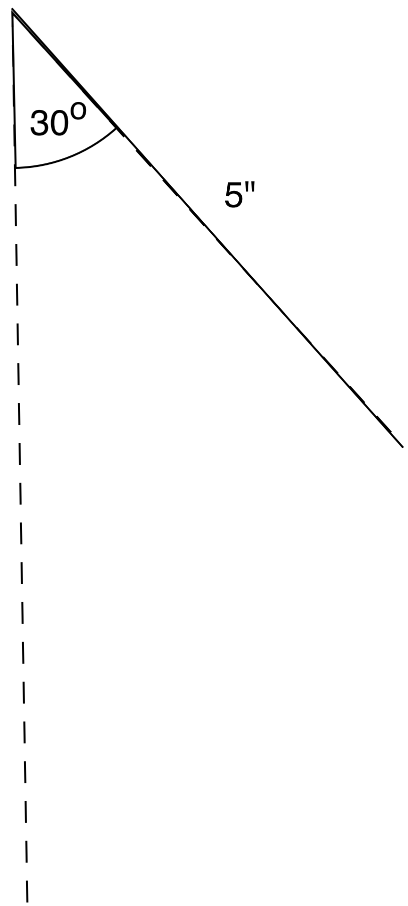
Card 7: Constructing Triangles



Suppose I choose angles 30°, 40° and a side 5" long.
Is there a way to make two triangles with these properties so they are not congruent?

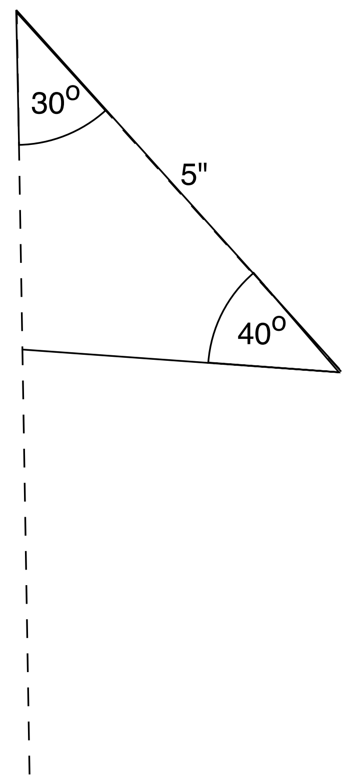
Card 7: Constructing Non-Congruent Triangles



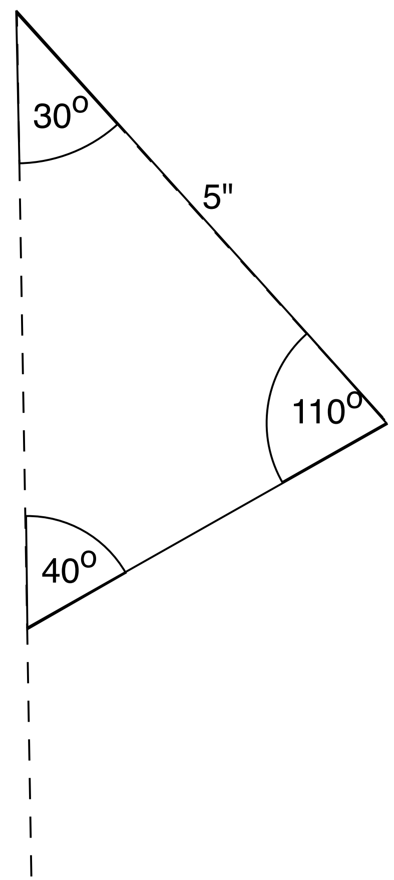


Where could I construct the 40° angle?

Card 7: Non-Congruent Triangles



Triangle 1



Triangle 2

Card Set: Must the Two Triangles be Congruent?

1. One side of Triangle A is the same length as one side of Triangle B.	2. Two sides of Triangle A are the same lengths as two sides of Triangle B.	3. Three sides of Triangle A are the same lengths as three sides of Triangle B.
4. One side of Triangle A is the same length as one side of Triangle B and one angle in Triangle A is the same size as one angle in Triangle B.	5. Two sides of Triangle A are the same lengths as two sides of Triangle B and one angle in Triangle A is the same size as one angle in Triangle B.	6. Three sides of Triangle A are the same lengths as three sides of Triangle B and one angle in Triangle A is the same size as one angle in Triangle B.
7. One side of Triangle A is the same length as one side of Triangle B and two angles in Triangle A are the same sizes as two angles in Triangle B.	**8.** Two sides of Triangle A are the same lengths as two sides of Triangle B and two angles in Triangle A are the same sizes as two angles in Triangle B.	9. Three sides of Triangle A are the same lengths as three sides of Triangle B and two angles in Triangle A are the same sizes as two angles in Triangle B.

Must the Two Triangles be Congruent?

For each card:

1. Draw examples of pairs of triangles A and B that have the properties stated in the card.
2. Decide whether the two triangles **must** be congruent. Record your decision at the bottom of the card.
3. If you decide that the triangles **do not** have to be congruent, draw examples and explain why.
4. If you decide that the triangles **must** be congruent, try to write a convincing proof.

Make sure to include Card 5 in your work, as the whole-class will discuss this statement.

Working Together

Take turns to select a card you have worked on.

When it is your turn:

- Glue the card in the middle of a blank sheet of paper.
- Explain your conclusion and how you reached that conclusion.
- Make sure everyone in your group understands your diagrams.
- Ask others in the group to share their reasoning.
- Try to to reach an agreed conclusion.
- Write an explanation together that is better than your individual explanations.

Make sure you discuss Card 5.

Card 5

5.

Two sides of Triangle A are the same lengths as two sides of Triangle B

and

one angle in Triangle A is the same size as one angle in Triangle B.

Jorge's Proof

Jorge. You have 2 triangles two sides of one equal to two sides of the other and the angle in between equal.

$AB = DE$ $BC = EF$ $\measuredangle ABC = \measuredangle DEF$.

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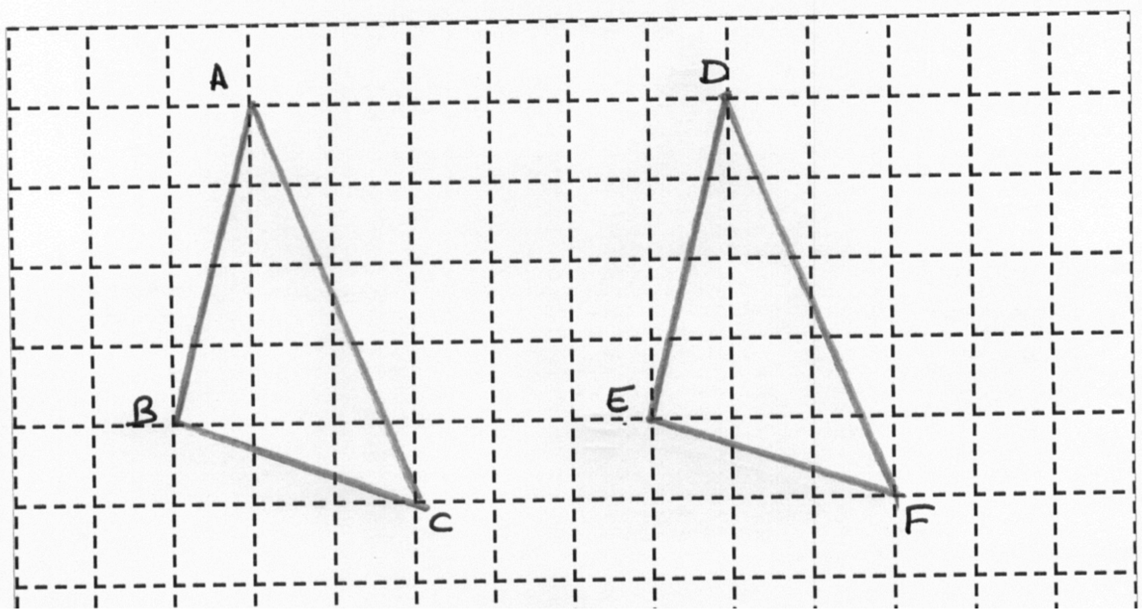
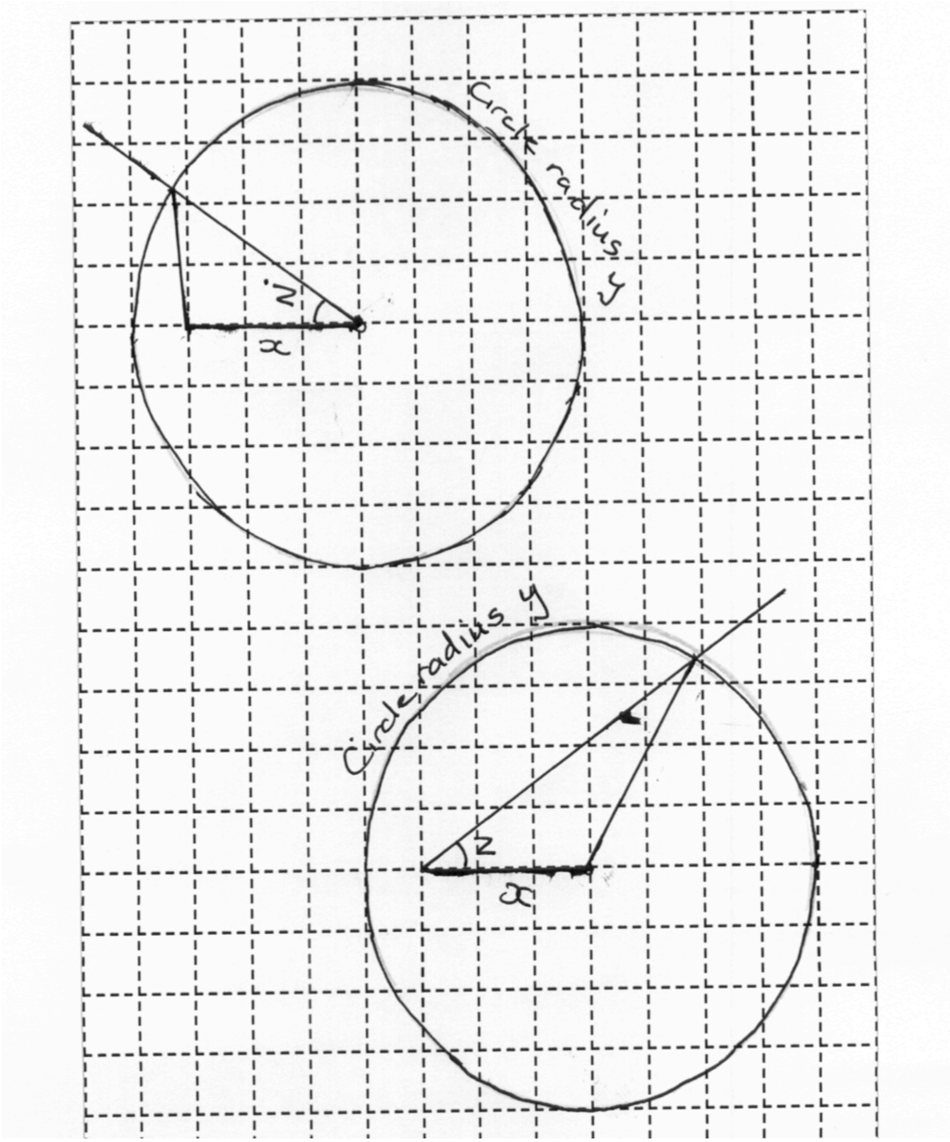


Kieran's Proof



Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
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