

Mathematics Assessment Project  
**CLASSROOM CHALLENGES**  
A Formative Assessment Lesson

# Representing Polynomials Graphically

Mathematics Assessment Resource Service  
University of Nottingham & UC Berkeley

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# Representing Polynomials Graphically

## MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to translate between graphs and algebraic representations of polynomials. In particular, this unit aims to help you identify and assist students who have difficulties:

- Recognizing the connection between the zeros of polynomials when suitable factorizations are available and graphs of the functions are defined by polynomials.
- Recognizing the connection between transformations of the graphs and transformations of the functions obtained by replacing  $f(x)$  by  $f(x + k)$ ,  $f(x) + k$ ,  $-f(x)$ ,  $f(-x)$ .

## COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

A-SSE: Interpret the structure of expressions.

A-APR: Understand the relationship between zeros and factors of polynomials.

F-IF: Analyze functions using different representations.

F-BF: Build new functions from existing functions.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 2 and 7:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

## INTRODUCTION

The lesson unit is structured in the following way:

- Before the lesson, students attempt the assessment task individually. You then review their work and formulate questions that will help them improve their solutions.
- During the lesson, students work collaboratively in pairs or threes, matching functions to their graphs and creating new examples. Throughout their work students justify and explain their decisions to peers. During a whole-class discussion, students explain their reasoning.
- Finally, students improve their solutions to the initial task and complete a second, similar task.

## MATERIALS REQUIRED

- Each student will need a mini-whiteboard, pen, and eraser, and a copy of *Cubic Graphs and Their Equations* and *Cubic Graphs and Their Equations (revisited)*.
- Each small group of students will need cut-up cards *Cubic Graphs*, *Cubic Functions* and *Statements to Discuss: True or False?*, a large sheet of poster paper, and a glue stick. You may want to enlarge the cards and/or copy them onto transparencies to be used on an overhead projector to support the whole-class discussion.

## TIME NEEDED

20 minutes before the lesson, a 90-minute lesson (or two 50-minute lessons), and 20 minutes in a follow-up lesson. Timings are approximate and will depend on the needs of your students.

## BEFORE THE LESSON

### Assessment task: *Cubic Graphs and Their Equations* (20 minutes)

Have students do this task, in class or for homework, a day or more before the formative assessment lesson. This will give you an opportunity to assess the work and identify students who have misconceptions or need other forms of help. You should then be able to target your help more effectively in the next lesson.

Give each student a copy of *Cubic Graphs and Their Equations*.

*Spend 20 minutes working individually, answering these questions.*

*Show all your work on the sheet.*

*Make sure you explain your answers clearly.*

It is important that, as far as possible, students answer the questions without assistance.

Students who sit together often produce similar answers and then when they come to work together on similar tasks, they have little to discuss. For this reason, we suggest that when

students do the assessment task, you ask them to move to different seats. Then at the beginning of the formative assessment lesson, allow them to return to their usual places. Experience has shown that this produces more profitable discussions.

### Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and difficulties. The purpose of this is to forewarn you of the issues that will arise during the lesson, so that you may prepare carefully.

We suggest that you do not score students' work. The research shows that this is counterproductive, as it encourages students to compare scores and distracts their attention from how they may improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a list of questions. Some suggestions for these are given in the *Common issues* table on the next page. We suggest that you make a list of your own questions, based on your students' work, using the ideas on the following page. We recommend you:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight appropriate questions for each student.

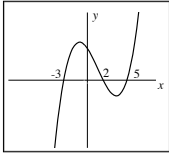
If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these questions on the board when you return the work to the students in the follow-up lesson.

**Cubic Graphs and Their Equations**

1. Write down an equation of a cubic function that would give a graph like the one shown here. It crosses the  $x$ -axis at  $(-3, 0)$ ,  $(2, 0)$ , and  $(5, 0)$ .

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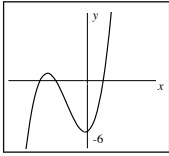
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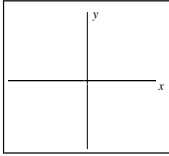
2. Write down an equation of a cubic function that would give a graph like the one shown here. It crosses the  $y$ -axis at  $(0, -6)$ .

.....

.....



3. On the axes, sketch a graph of the function  $y = (x+1)(x-4)^2$ . You do not need to plot it accurately! Show where the graph crosses the  $x$ - and  $y$ -axes.



4. Write down the equation of the graph you get after you:

(i) Reflect  $y = (x+1)(x-4)^2$  over the  $x$ -axis:

.....

(ii) Reflect  $y = (x+1)(x-4)^2$  over the  $y$ -axis:

.....

(iii) Horizontally translate  $y = (x+1)(x-4)^2$  through +2 units:

.....

(iv) Vertically translate  $y = (x+1)(x-4)^2$  through +3 units:

.....

**Common issues:****Suggested questions and prompts:**

<p><b>Writes an equation that is not cubic (Q1, 2)</b></p>	<ul style="list-style-type: none"> <li>• Does your equation give a straight or curved graph? How do you know?</li> <li>• From your equation, when <math>y = 0</math> (or <math>x = 0</math>), what must <math>x</math> (or <math>y</math>) be? Does this agree with the graph?</li> <li>• Will there be positive or negative signs inside the parentheses?</li> <li>• How do the number of roots equate to the number of parentheses in the function?</li> <li>• What happens to the <math>y</math> values when <math>x</math> is really large? How can this be represented in the function?</li> </ul>
<p><b>Writes a cubic equation, but gets the signs wrong (Q1, 2)</b></p> <p>For example: The student writes (Q1):  <math>y = (x - 3)(x + 2)(x + 5)</math>.</p>	<ul style="list-style-type: none"> <li>• Where does the equation tell you that the graph crosses the <math>x</math>-axis?</li> <li>• What value for <math>y</math> do you get when <math>x = 2</math>? Does this correspond with the graph?</li> </ul>
<p><b>Writes an incorrect equation (Q2)</b></p> <p>For example: The student correctly takes account of the <math>y</math>-intercept, but not the <math>x</math>-intercepts: <math>y = x^3 - 6</math>.</p> <p>Or: The student correctly takes account of the <math>x</math>-intercepts but ignores the <math>y</math>-intercepts:  <math>y = (x + 3)(x + 4)(x - 2)</math>.</p>	<ul style="list-style-type: none"> <li>• Will this cross the <math>x</math>-axis in approximately the right places?</li> <li>• Will this cross the <math>y</math>-axis in approximately the right places?</li> </ul>
<p><b>Plots the graph rather than sketches it (Q3)</b></p>	<ul style="list-style-type: none"> <li>• Can you get an idea of the shape of the graph without plotting points?</li> <li>• Can you get an idea of the shape of the graph by looking at where the graph crosses the axes?</li> <li>• What happens when <math>x</math> is very large?</li> </ul>
<p><b>Incorrectly sketches the graph (Q3)</b></p> <p>For example: The student sketches a graph with three roots.</p> <p>Or: The student sketches a graph with <math>x</math>-intercepts at <math>x = 1</math> and <math>x = 4</math>.</p> <p>Or: When sketching the graph the student does not use an approximate scale.</p>	<ul style="list-style-type: none"> <li>• What does the function tell you about the number of roots?</li> <li>• To help sketch the graph, what numbers can you substitute into the function?</li> <li>• Check your graph is correct. What are the values for <math>x</math> when <math>y</math> is zero?</li> <li>• Check you have placed your <math>x</math>- and <math>y</math>-intercepts in approximately the correct position for the scale you've used.</li> </ul>
<p><b>Attempts to draw a reflection or translation of the graph and then fit a function to it (Q4)</b></p>	<ul style="list-style-type: none"> <li>• What happens to the point <math>(3, 2)</math> after it is: <ul style="list-style-type: none"> <li>(i) Reflected over the <math>x</math>-axis?</li> <li>(ii) Reflected over the <math>y</math>-axis?</li> <li>(iii) Translated +2 units horizontally?</li> <li>(iv) Translated +3 vertically?</li> </ul> </li> <li>• So what happens to every <math>x</math> value (<math>y</math> value) in the function?</li> <li>• After the translation, where will the roots be?</li> </ul>

## SUGGESTED LESSON OUTLINE

In this lesson students should **not** use a graphing calculator.

### Introduction: sketching and interpreting graphs of polynomials (25 minutes)

Give each student a mini-whiteboard, pen, and eraser. Maximize participation in the discussion by asking all students to show solutions on their mini-whiteboards.

Throughout this introduction encourage students to justify their responses. Try not to correct responses, but encourage students to challenge each others' explanations.

*Show me a sketch of the graph of  $y = 2x - 8$ , marking just the  $x$ - and  $y$ -intercepts.*

*Now do the same for  $2y = 5x - 10$ .*

Ensure that students know how to find the  $y$ -intercept by putting  $x$  equal to zero and the  $x$ -intercept(s) by equating  $y$  to zero. Repeat this for a few quadratic graphs.

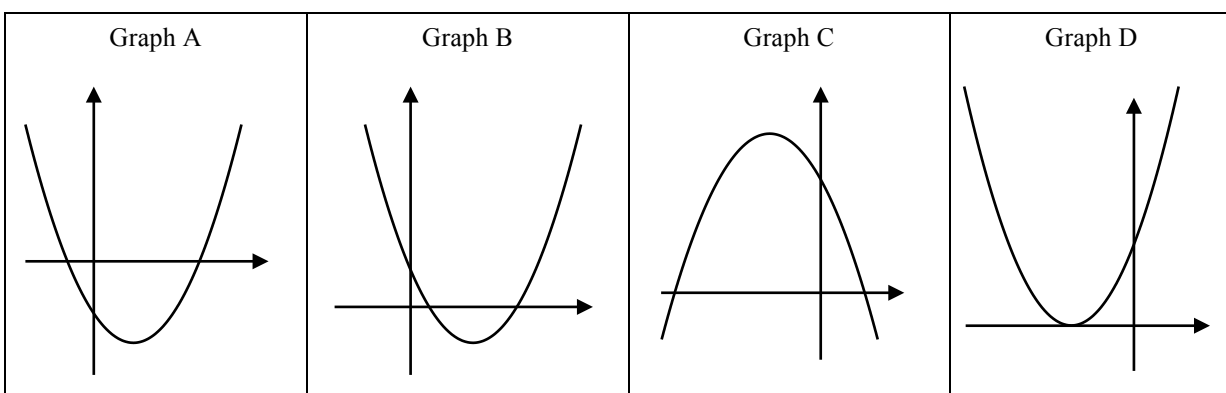
*Show me a sketch of the graph of  $y = (x - 3)(x + 4)$ , marking the  $x$ - and  $y$ -intercepts.*

Some students may sketch a graph with one negative and one positive  $x$ -intercept, but position them incorrectly. This may be because they assume the intercepts are at  $x = -3$  and  $x = 4$ , or they do not apply an approximate scale correctly. To find the  $y$ -intercept some students may multiply out the parentheses, whilst others will substitute  $x = 0$  into the equation. You may also find some students sketch graphs with two  $y$ -intercepts and one  $x$ -intercept. Some students may figure out the coordinates of the vertex. This would improve the accuracy of their sketch, but it may also derail the goals of this lesson, as it is a very time consuming activity.

Throughout this introduction your task is to investigate students' strategies and solutions.

Display Slide P-1 of the projector resource.

*Show me a possible equation that fits ... (Graph A / Graph B / Graph C / Graph D).  
How can you check your answer?*



*Graph A could be  $y = (x + 1)(x - 4)$  because it might intersect the  $x$ -axis at about  $(-1, 0)$  and  $(+4, 0)$ .*

*Graph B could be  $y = (x - 1)(x - 3)$  because it might intersect the  $x$ -axis at about  $(+1, 0)$  and  $(+3, 0)$ .*

*Graph C could be  $y = -(x - 1)(x + 5)$  because it might intersect the  $x$ -axis at about  $(+1, 0)$  and  $(-5, 0)$  and when  $x$  is large (say 100), then  $y$  is negative.*

*Graph D could be  $y = (x + 3)^2$  because it only intersects the  $x$ -axis at one point, so it must have a repeated root.*

In this way, students should learn to pay particular attention to the intercepts and the sign of  $y$  when  $x$  is very large (or very small).

For each correct suggestion, ask students to state where the graph crosses the  $y$ -axis and give their reasons.

Students should now be in a good position to explore cubics. Ask students to sketch  $y = (x - 1)(x - 2)(x - 3)$  on their mini-whiteboards, using the same ideas (as they used with the quadratic functions) of finding the intercepts first. Check that everyone understands by asking them to explain their graphs. Repeat this using additional examples using both  $+$  and  $-$  inside the parentheses.

### Collaborative small-group work (20 minutes)

Organize students into groups of two or three. For each group, provide a cut-up copy of the *Cubic Graphs* and *Cubic Functions* cards and a large sheet of paper for making a poster. Do not distribute the glue sticks yet.

Display Slide P-2 of the projector resource and explain how students are to work together:

Working Together
<ol style="list-style-type: none"><li>1. Take turns to match a function to its graph.</li><li>2. As you do this, label the graph to show the intercepts on the <math>x</math>- and <math>y</math>-axes.</li><li>3. If you match two cards, explain how you came to your decision.</li><li>4. If you don't agree or understand, ask your partner to explain their reasoning.</li><li>5. You <b>all</b> need to agree on and explain the matching of every card.</li><li>6. You may find that more than one function will fit some graphs!</li><li>7. If you have functions left over, sketch graphs on the blank cards to match these functions.</li></ol>

The purpose of this structured work is to encourage students to engage with each other's explanations and take responsibility for each other's understanding.

*Darren matched these cards. Marla, can you explain Darren's thinking?*

If you find the student is unable to answer this question, ask them to discuss the work further. Explain that you will return in a few minutes to ask a similar question.

During small group work, try to listen and support students' thinking and reasoning. Note difficulties that emerge for more than one group.

Are students matching a function to a graph or a graph to a function?

Do students thoughtfully substitute values for  $x$  into a function?

Do students use the zeros of the functions effectively?

Are students able to use very large or very small values of  $x$  to discriminate between  $y = f(x)$  and  $y = -f(x)$ ?

Are students multiplying out the parentheses? And if so, why?

Do students find the coordinates of the vertex? And if so, why? (The coordinates of the vertex are not needed for this task.)

Do students attend carefully to the signs in the function?

Do students recognize repeated roots in a function?

Are students making assumptions? For example, do students assume that if there is one positive  $x$ -intercept and one negative  $x$ -intercept the vertex will lie on the  $y$ -axis?

You will be able to use this information in the whole-class discussion.

Try not to make suggestions that move students towards a particular strategy. Instead, ask questions to help students to reason together. Encourage students to use an efficient method. You may want to use the questions in the *Common issues* table to help address misconceptions.

It is not essential that everyone completes this activity, but it is important that everyone has matched the six graph cards to some functions. Function cards 7 and 11 are the same function, but there is no graph card for this function. Students that are struggling with sketching a graph for function card 7 could be encouraged to look at how it relates to the other functions given (i.e. function 7 is a translation of function 8/graph F). They may also choose to employ an algebraic method to help with this.

Once you are satisfied students have matched enough cards, give them a glue stick and ask them to glue their cards onto the paper to make a poster.

### **Extending the lesson over two days**

If you are taking two days to complete the unit you might want to end the first lesson here. Before the end of the lesson, make sure that students have glued onto their poster paper **all** the cards they have matched so far. Then, at the start of the second day, allow the students time to familiarize themselves with their posters, before they share their work with another group.

### **Sharing work (15 minutes)**

When students have completed the task, ask them to check their work against that of a neighboring group:

*Check to see which matches are different from your own.*

*If there are differences, ask for an explanation. If you still don't agree, explain your own thinking.*

*You may then need to consider whether to make any changes to your own work.*

*It is important that everyone in both groups understands the math. You are responsible for each other's learning.*

### **Whole-class discussion (30 minutes)**

To promote an interactive discussion between all students ask them to write solutions on their mini-whiteboards.

Organize a whole-class discussion about the different strategies used to match the cards. The intention is that you work with the students to make the math of the lesson explicit.

First select a set of cards that most groups matched correctly. This approach may encourage good explanations. Then select one or two cards that most groups found difficult to match.

Once one group has justified their choice for a particular match, ask other students to contribute ideas of alternative approaches and their views on which reasoning method was easier to follow. The intention is that you focus on getting students to understand and share their **reasoning**, not just checking that everyone produced the right answers. Use your knowledge of the students' individual and group work to call on a wide range of students for contributions.

*How did you decide to match this card?*

*Can someone else put that into their own words?*

You may want to draw on the questions in the *Common issues* table to support your own questioning.

You may also choose to begin to generalize the learning by asking students to pair up two sets of cards that have some similar graphical or/and algebraic features. After a few minutes working individually on the task, they are to discuss it with a partner.

Once they have completed the task, ask each group of students to report back on their pair of statements, giving examples and full explanations. This should reinforce the important lessons learned. You may want to support the discussion with questions from the *Common issues* table.

*What do these two functions have in common? What are their differences?*

*How are these similarities and differences represented in the graph?*

*Show me two graphs that are reflections of each other.*

*Describe the reflection.*

*How is this reflection represented in the function?*

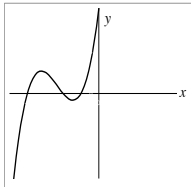
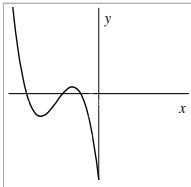
*Show me two graphs that represent a translation of one graph onto another.*

*Describe the translation.*

*How is this translation represented in the function?*

Follow-up suggestions by prompting students to be specific and if appropriate, move towards more general representations.

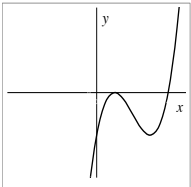
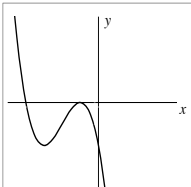
Below are four possible answers:

<p><b>Graph A</b></p> 	<p><b>Function 2</b></p> $y = (x + 1)(x + 2)(x + 4)$	<p><b>Graph E</b></p> 	<p><b>Function 3</b></p> $y = -(x + 1)(x + 2)(x + 4)$
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*Graph A is a reflection of Graph E over the  $x$ -axis.*

*How can you tell from the equation that this is a reflection? [The negative sign in front of the function.]*

*If Graph A is represented by  $y = f(x)$ , then how can we represent Graph E? [ $y = -f(x)$ .]*

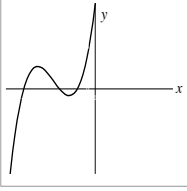
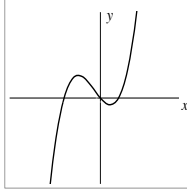
<p><b>Graph C</b></p> 	<p><b>Function 4</b></p> $y = (x - 1)^2(x - 4)$	<p><b>Graph F</b></p> 	<p><b>Function 8</b></p> $y = -(x + 1)^2(x + 4)$
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*Graph C is a reflection of Graph F over the  $y$ -axis.*

*How can you tell from the equation that this is a reflection? [Replace  $x$  by  $-x$  in the equation for Graph C and simplify, noting that  $(-x - 1)^2 = (-1)(x + 1)(-1)(x + 1) = (x + 1)^2$  and that  $(-x - 4) = -(x + 4)$ .]*

*If Graph C is represented by  $y = f(x)$ , then how can we represent Graph F? [ $y = f(-x)$ .]*



<b>Graph A</b>	<b>Function 2</b>	<b>Graph B</b>	<b>Function 1</b>
	$y = (x + 1)(x + 2)(x + 4)$		$y = x(x - 1)(x + 2)$

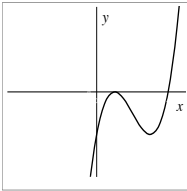
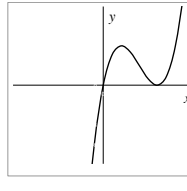
Graph A is a horizontal translation of Graph B.

How far does Graph A have to translate horizontally to get Graph B? [+2 units.]

How can you tell? [The middle root on Graph A is at  $x = -2$ , on Graph B it is at the origin.]

How can we transform the equation for Graph A to get the equation for Graph B? [Replace each instance of  $x$  in the equation by  $x - 2$ .]

If Graph A is represented by  $y = f(x)$ , then how can we represent Graph B? [ $y = f(x - 2)$ .]

<b>Graph C</b>	<b>Function 4</b>	<b>Graph D</b>	<b>Function 6</b>
	$y = (x - 1)^2(x - 4)$		<b>Function 9</b> $y = (x - 1)^2(x - 4) + 4$

Graphs D is a vertical translation of Graph C.

How far does Graph C have to vertically translate to get Graph D? [+4 units.]

How can you tell? [The y-intercept on Graph C is at  $y = -4$ , on Graph D it is at the origin.]

How can we transform the equation for Graph C to get the equation for Graph D? [Simply add 4 onto the end of the function.]

If Graph C is represented by  $y = f(x)$ , then how can we represent Graph D? [ $y = f(x) + 4$ .]

### (Optional) Extension activity: small-group work 10 minutes

If students have done well, you may wish to give each pair/group of students a copy of *Statements to Discuss: True or False*. This sheet attempts to further develop and extend students' understanding.

Explain to students what they need to do:



*Choose one or two pairs of statements.*

*Decide which statement is true and which is false.*

*Be prepared to justify your answers.*

Give students 5 minutes to work on each pair of statements.

You may wish to allocate particular statements to specific students to provide an appropriate challenge.

A1 $f(x) = x^3 - 2x^2 - 9x + 18$ $f(2) = 0 \Rightarrow (x + 2)$ is a factor of $f(x)$	A2 $f(x) = 3x^3 - 9x - 6$ $f(2) = 0 \Rightarrow (x - 2)$ is a factor of $f(x)$
B1  A possible equation for this graph is: $y = (x + 1)(x - 2)^2$	B2  A possible equation for this graph is: $y = (x - 1)(x + 2)^2$
C1 If $f(x)$ is a cubic function and $f(1) = 0$ , $f(3) = 0$ and $f(4) = 0$ , then $f(5) = 8$	C2 If $f(x)$ is a cubic function and $f(1) = 0$ , $f(3) = 0$ and $f(4) = 0$ , then $f(5)$ could be any number but zero.
D1 $f(x) = (x - 2)^2(x - 7)$ $g(x) = (x - 2)^2(7 - x)$ $f(x)$ is a reflection of $g(x)$ over the y axis	D2 $f(x) = x(x - 2)^2$ $g(x) = -x(-x - 2)^2$ $f(x)$ is a reflection of $g(x)$ over the y axis
E1 $f(x) = g(x + 2) \Rightarrow$ $g(x)$ is a horizontal translation of $f(x)$	E2 $f(x) = g(x) + 2 \Rightarrow$ $g(x)$ is a horizontal translation of $f(x)$

**Follow-up lesson: reviewing the assessment task (20 minutes)**

Return to the students their original assessment *Cubic Graphs and Their Equations*.

If you have not added questions to individual pieces of work, then write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

*Read through your original solution to the task and the questions I have written (on your script/on the board).*

*Spend a few minutes answering these questions and revise your response. You may want to make notes on your mini-whiteboard.*

To help students see their own progress, ask them to complete the task using a different color pen or give them a second blank copy of the task.

Now give students a copy of the task *Cubic Graphs and Their Equations (revisited)*.

*Use what you have learned to complete the new assessment task. Show all your work on the sheet and make sure you explain your answers really clearly.*

If students struggled with the original assessment task, you may feel it more appropriate for them to just spend their time revisiting *Cubic Graphs and Their Equations* rather than attempting *Cubic Graphs and Their Equations (revisited)* as well.

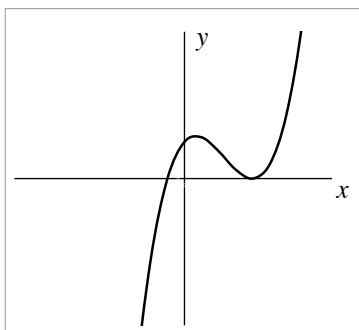
## SOLUTIONS

### Assessment task: *Cubic Graphs and Their Equations*

1. A suitable cubic function will take the form:  $y = A(x + 3)(x - 2)(x - 5)$  where  $A$  is any real positive number.
2. A suitable cubic function will take the form:  $y = A(x + a)(x + b)(x - c)$  where  $A$  is any positive real number, and  $a, b, c$  are positive numbers, such that  $a > b > c$  and  $Aabc = 6$ .  
Specific examples are:

$$y = (x + 3)(x + 2)(x - 1), \text{ or } y = \frac{(x + 4)(x + 3)(x - 2)}{4}$$

3. A suitable sketch will show the  $x$ -axis intercepts at  $(-1, 0)$  and  $(4, 0)$ , the  $y$ -axis intercept at  $(0, 16)$  and be of the following shape:



4. (i) When reflected in the  $x$ -axis, the equation will become:  $y = -(x + 1)(x - 4)^2$

- (ii) When reflected in the  $y$ -axis, the equation will become:

$$y = (-x + 1)(-x - 4)^2 = (-x + 1)(x + 4)^2$$

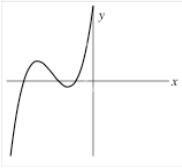
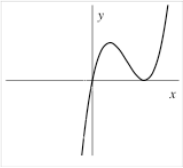
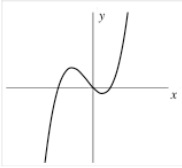
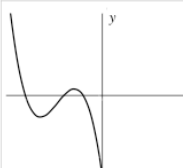
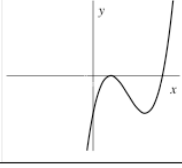
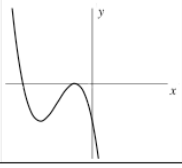
- (iii) When horizontally translated +2 units, the equation will become:

$$y = ((x - 2) + 1)((x - 2) - 4)^2 = (x - 1)(x - 6)^2$$

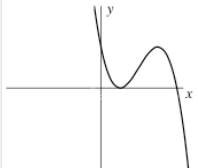
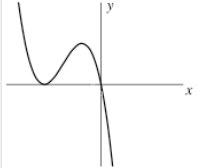
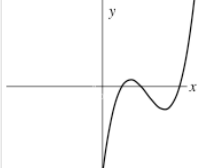
- (iv) When vertically translated +3 units, the equation will become:  $y = (x + 1)(x - 4)^2 + 3$

**Lesson task:**

The correct matching of the graphs and functions is shown below:

<p><b>Graph A</b></p> 	<p><b>Function 2</b></p> $y = (x+1)(x+2)(x+4)$	<p><b>Graph D</b></p> 	<p><b>Function 6</b></p> $y = x(x-3)^2$ <p><b>Function 9</b></p> $y = (x-1)^2(x-4) + 4$
<p><b>Graph B</b></p> 	<p><b>Function 1</b></p> $y = x(x-1)(x+2)$	<p><b>Graph E</b></p> 	<p><b>Function 3</b></p> $y = -(x+1)(x+2)(x+4)$
<p><b>Graph C</b></p> 	<p><b>Function 4</b></p> $y = (x-1)^2(x-4)$	<p><b>Graph F</b></p> 	<p><b>Function 8</b></p> $y = -(x+1)^2(x+4)$

The missing graphs are:

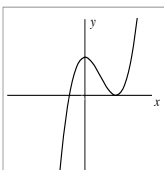
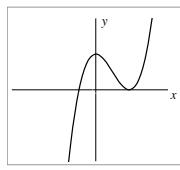
<p><b>Function 5</b></p> $y = -(x-1)^2(x-4)$	
<p><b>Function 7</b></p> $y = -(x+1)^2(x+4) + 4$ <p><b>Function 11</b></p> $y = -x(x+3)^2$	
<p><b>Function 10</b></p> $y = (x-1)(x-2)(x-4)$	

**Extension activity: Statements to Discuss: True or False?**

<p>A1</p> $f(x) = x^3 - 2x^2 - 9x + 18$ $f(2) = 0 \Rightarrow (x + 2) \text{ is a factor of } f(x)$	<p>A2</p> $f(x) = 3x^3 - 9x - 6$ $f(2) = 0 \Rightarrow (x - 2) \text{ is a factor of } f(x)$
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**A1 is false; A2 is true.**

If  $f(2) = 0$ ,  $(x - 2)$  *must* be a factor of  $f(x)$ .

<p>B1</p>  <p>A possible equation for this graph is:</p> $y = (x + 1)(x - 2)^2$	<p>B2</p>  <p>A possible equation for this graph is:</p> $y = (x - 1)(x + 2)^2$
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**B1 is true; B2 is false.**

There are also other possible equations for the function, such as  $y = 2(x + 1)(x - 2)^2$ .

B2 is clearly false as the repeated root must be positive.

<p>C1</p> <p>If <math>f(x)</math> is a cubic function and <math>f(1) = 0</math>, <math>f(3) = 0</math> and <math>f(4) = 0</math>, then <math>f(5) = 8</math></p>	<p>C2</p> <p>If <math>f(x)</math> is a cubic function and <math>f(1) = 0</math>, <math>f(3) = 0</math> and <math>f(4) = 0</math>, then <math>f(5)</math> could be any number but zero</p>
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**C1 is true; C2 is false.**

The conditions mean that the function must take the form:  $f(x) = A(x - 1)(x - 3)(x - 4)$ , where  $A$  is a non-zero constant. Now  $f(5) = 8A$ , so it can take any value apart from zero.

<p>D1</p> $f(x) = (x - 2)^2(x - 7)$ $g(x) = (x - 2)^2(7 - x)$ <p><math>f(x)</math> is a reflection of <math>g(x)</math> over the <math>y</math> axis</p>	<p>D2</p> $f(x) = x(x - 2)^2$ $g(x) = -x(-x - 2)^2$ <p><math>f(x)</math> is a reflection of <math>g(x)</math> over the <math>y</math> axis</p>
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**D1 is false; D2 is true.**

D1 is a reflection over the  $x$ -axis.

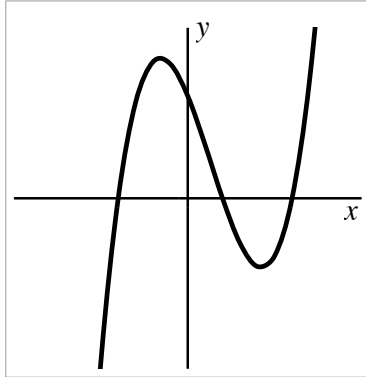
<p>E1</p> $f(x) = g(x + 2) \Rightarrow$ <p><math>g(x)</math> is a horizontal translation of <math>f(x)</math></p>	<p>E2</p> $f(x) = g(x) + 2 \Rightarrow$ <p><math>g(x)</math> is a horizontal translation of <math>f(x)</math></p>
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**E1 is true; E2 is false.**

E2 is a vertical translation.

**Assessment task: Cubic Graphs and Their Equations (revisited)**

1. A suitable cubic function will take the form:  $y = Ax^2(x-6)$  or  $y = Ax(x-6)^2$  where  $A$  is any real number, positive or negative.
2. A suitable sketch will show the  $x$ -axis intercepts at  $(-2, 0)$ ,  $(1, 0)$ , and  $(3, 0)$  the  $y$ -axis intercept at  $(0, 6)$  and be of the following shape:



When reflected over the  $y$ -axis, the equation will become:

$$y = (-x+2)(-x-3)(-x-1) = (2-x)(x+3)(x+1)$$

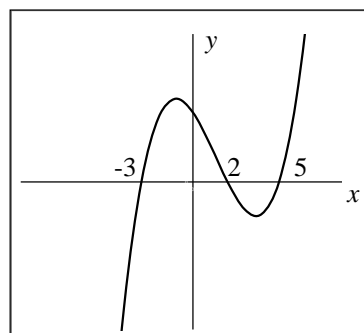
3. A suitable cubic function will take the form:  $y = A(x+a)(x+b)(x+c)$  where  $A$  is any real positive number, and  $a, b, c$  are positive numbers, such that  $a > b > c$  and  $Aabc = 8$ . Specific examples are:

$$y = (x+4)(x+2)(x+1), \text{ or } y = \frac{(x+8)(x+2)(x+1)}{2}$$

4. (i) A vertical translation through +4 units.  
(ii) A reflection over the  $x$ -axis.  
(iii) A horizontal translation through -3 units  $y = ((x+3)-2)((x+3)+3)^2 = (x+1)(x+6)^2$

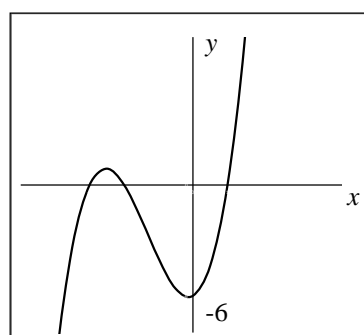
# Cubic Graphs and Their Equations

1. Write down an equation of a cubic function that would give a graph like the one shown here. It crosses the  $x$ -axis at  $(-3, 0)$ ,  $(2, 0)$ , and  $(5, 0)$ .



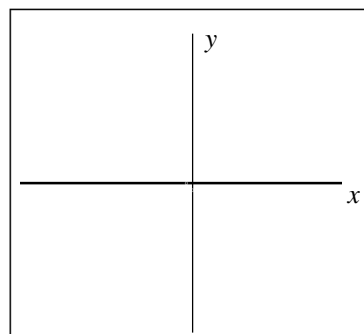
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2. Write down an equation of a cubic function that would give a graph like the one shown here. It crosses the  $y$ -axis at  $(0, -6)$ .



.....  
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3. On the axes, sketch a graph of the function  $y = (x+1)(x-4)^2$ .  
 You do not need to plot it accurately!  
 Show where the graph crosses the  $x$ - and  $y$ -axes.



4. Write down the equation of the graph you get after you:

(i) Reflect  $y = (x+1)(x-4)^2$  over the  $x$ -axis:

.....

(ii) Reflect  $y = (x+1)(x-4)^2$  over the  $y$ -axis:

.....

(iii) Horizontally translate  $y = (x+1)(x-4)^2$  through +2 units:

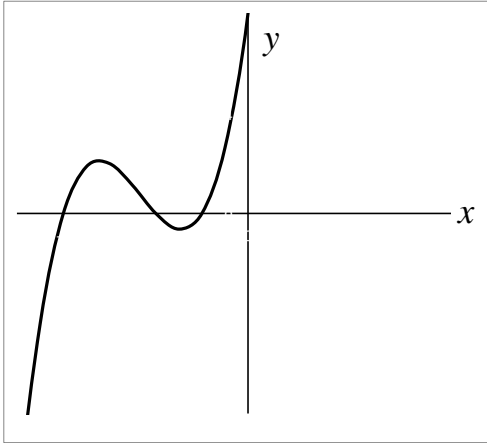
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(iv) Vertically translate  $y = (x+1)(x-4)^2$  through +3 units:

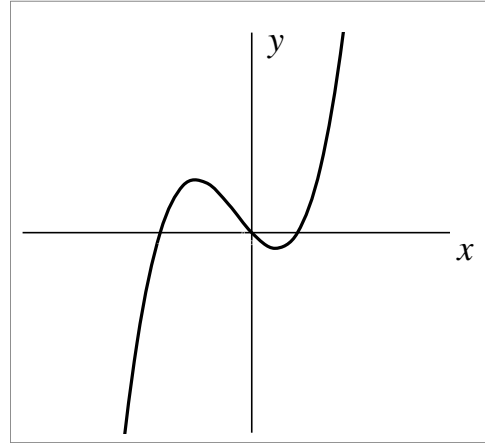
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# Cubic Graphs

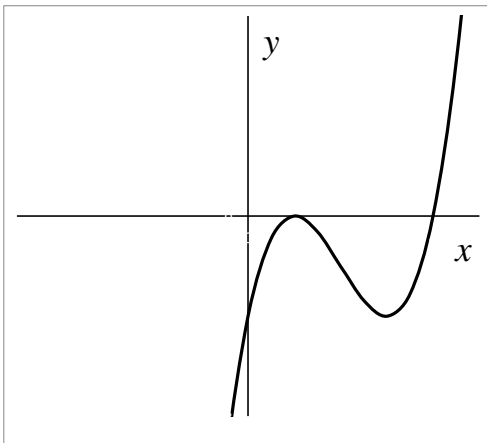
Graph A



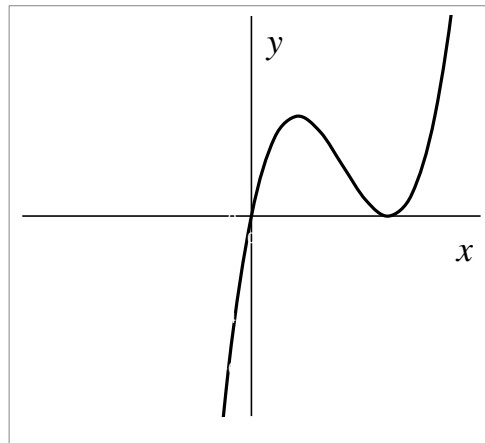
Graph B



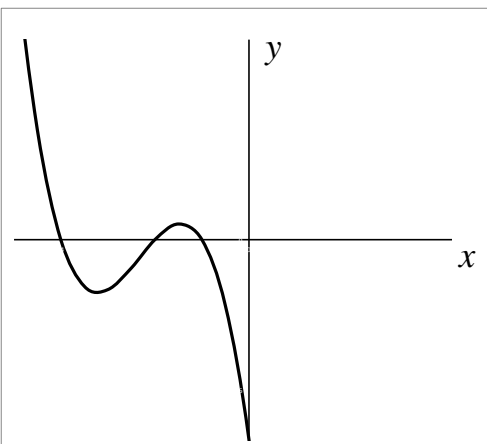
Graph C



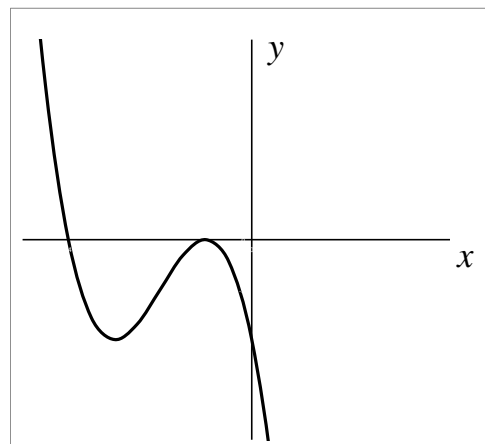
Graph D



Graph E



Graph F

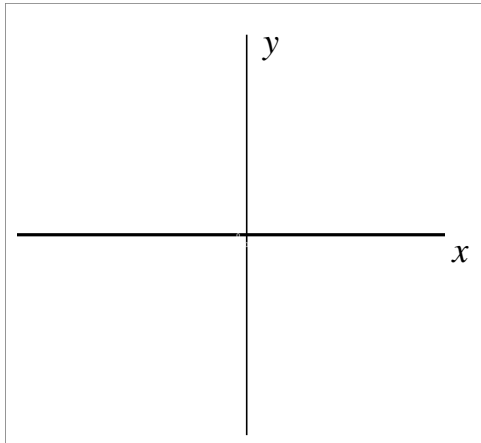




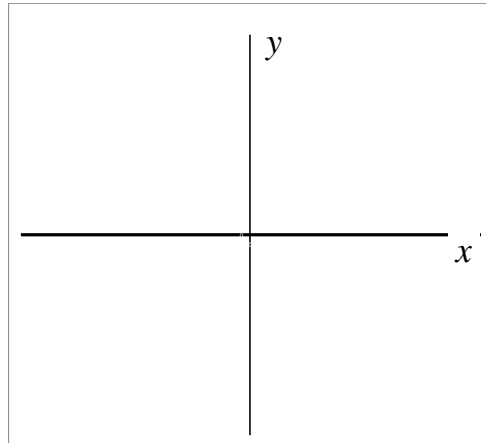
## Cubic Graphs (continued)

Complete these graphs for the remaining functions:

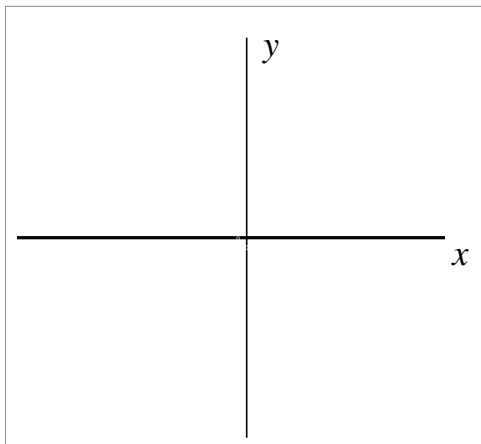
**Graph G**



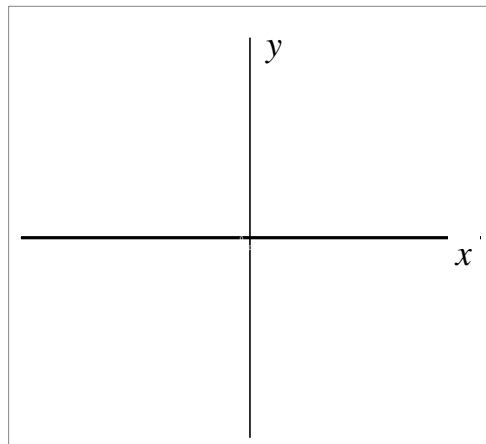
**Graph H**



**Graph I**



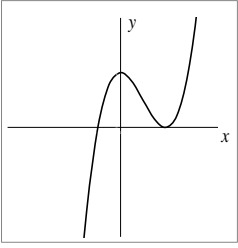
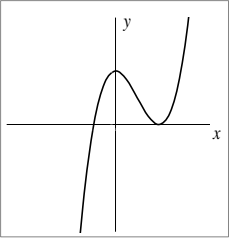
**Graph J**



## Cubic Functions

1	$y = x(x - 1)(x + 2)$
2	$y = (x + 1)(x + 2)(x + 4)$
3	$y = -(x + 1)(x + 2)(x + 4)$
4	$y = (x - 1)^2(x - 4)$
5	$y = -(x - 1)^2(x - 4)$
6	$y = x(x - 3)^2$
7	$y = -(x + 1)^2(x + 4) + 4$
8	$y = -(x + 1)^2(x + 4)$
9	$y = (x - 1)^2(x - 4) + 4$
10	$y = (x - 1)(x - 2)(x - 4)$
11	$y = -x(x + 3)^2$

## Statements to Discuss: True or False?

<p>A1</p> $f(x) = x^3 - 2x^2 - 9x + 18$ $f(2) = 0 \Rightarrow (x+2) \text{ is a factor of } f(x)$	<p>A2</p> $f(x) = 3x^3 - 9x - 6$ $f(2) = 0 \Rightarrow (x-2) \text{ is a factor of } f(x)$
<p>B1</p> <div style="display: flex; align-items: center;">  <div style="flex-grow: 1;"> <p>A possible equation for this graph is:</p> <math display="block">y = (x+1)(x-2)^2</math> </div> </div>	<p>B2</p> <div style="display: flex; align-items: center;">  <div style="flex-grow: 1;"> <p>A possible equation for this graph is:</p> <math display="block">y = (x-1)(x+2)^2</math> </div> </div>
<p>C1</p> <p>If <math>f(x)</math> is a cubic function and <math>f(1) = 0</math>, <math>f(3) = 0</math> and <math>f(4) = 0</math>, then <math>f(5) = 8</math></p>	<p>C2</p> <p>If <math>f(x)</math> is a cubic function and <math>f(1) = 0</math>, <math>f(3) = 0</math> and <math>f(4) = 0</math>, then <math>f(5)</math> could be any number but zero.</p>
<p>D1</p> $f(x) = (x-2)^2(x-7)$ $g(x) = (x-2)^2(7-x)$ <p><math>f(x)</math> is a reflection of <math>g(x)</math> over the y axis</p>	<p>D2</p> $f(x) = x(x-2)^2$ $g(x) = -x(-x-2)^2$ <p><math>f(x)</math> is a reflection of <math>g(x)</math> over the y axis</p>
<p>E1</p> $f(x) = g(x+2) \Rightarrow$ <p><math>g(x)</math> is a horizontal translation of <math>f(x)</math></p>	<p>E2</p> $f(x) = g(x) + 2 \Rightarrow$ <p><math>g(x)</math> is a horizontal translation of <math>f(x)</math></p>

# Cubic Graphs and Their Equations (revisited)

1. A cubic function has just two  $x$ -intercepts, one at  $x = 0$  and the other at  $x = 6$ .

Write down at least three possible equations with these intercepts:

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.....

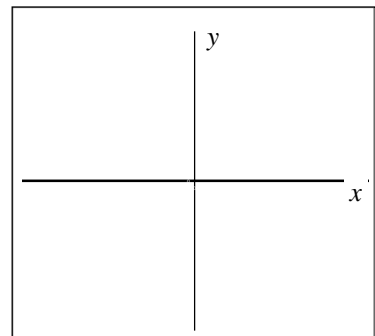
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2. On the axes, sketch a graph of the function  $y = (x+2)(x-3)(x-1)$

You do not need to plot it accurately!

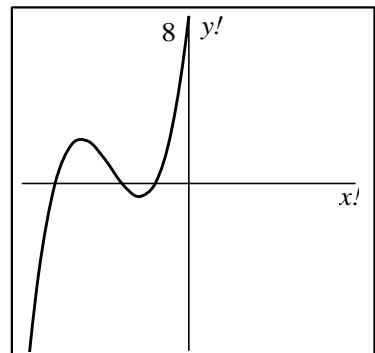
Show where the graph crosses the  $x$ - and  $y$ -axes.

Write down the equation of the graph after you reflect it over the  $y$ -axis.



.....

3. Write down an equation of a cubic function that would give a graph like the one shown here. It crosses the  $y$ -axis at  $(0, 8)$ .



.....

.....

4. The function  $y = (x-2)(x+3)^2$  is transformed to:

(i)  $y = (x-2)(x+3)^2 + 4$ .

Describe in words how the function has been transformed.

.....

(ii)  $y = -(x-2)(x+3)^2$ .

Describe in words how the function has been transformed.

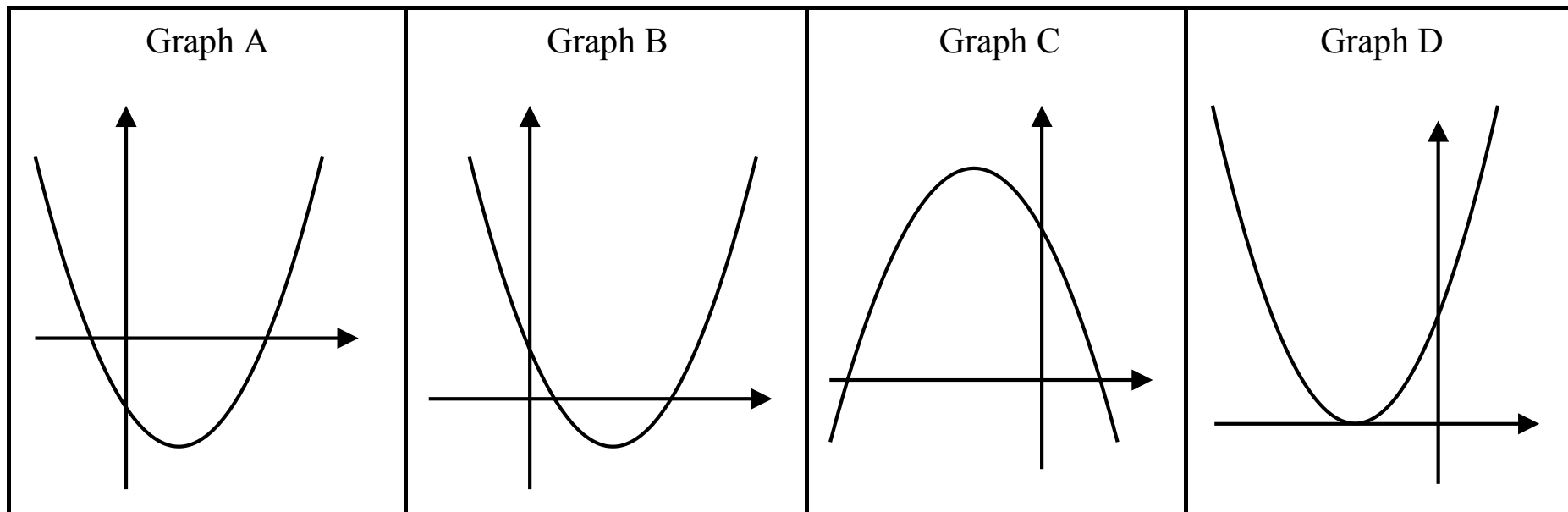
.....

(iii)  $y = (x+1)(x+6)^2$ .

Describe in words how the function has been transformed.

.....

# Show me an equation that fits ...



# Working Together

1. Take turns to match a function to its graph.
2. As you do this, label the graph to show the intercepts on the  $x$ - and  $y$ -axes.
3. If you match two cards, explain how you came to your decision.
4. If you don't agree or understand, ask your partner to explain their reasoning.
5. You **all** need to agree on and explain the matching of every card.
6. You may find that more than one function will fit some graphs!
7. If you have functions left over, sketch graphs on the blank cards to match these functions.

# True or False?

A1  $f(x) = x^3 - 2x^2 - 9x + 18$

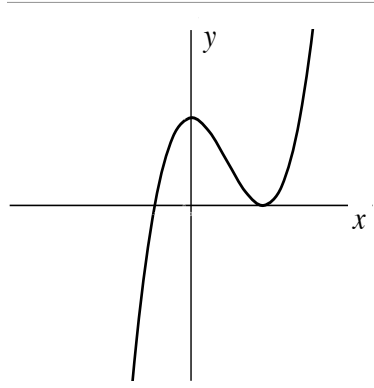
$f(2) = 0 \Rightarrow (x + 2)$  is a factor of  $f(x)$

A2  $f(x) = 3x^3 - 9x - 6$

$f(2) = 0 \Rightarrow (x - 2)$  is a factor of  $f(x)$

# True or False?

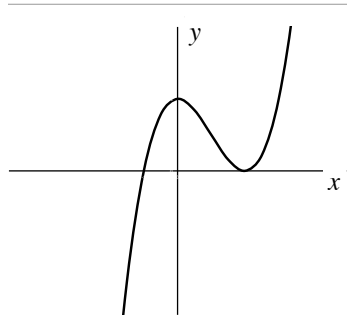
B1



A possible equation for this graph is:

$$y = (x + 1)(x - 2)^2$$

B2



A possible equation for this graph is:

$$y = (x - 1)(x + 2)^2$$



# True or False?

- C1      If  $f(x)$  is a cubic function  
and  $f(1) = 0$ ,  $f(3) = 0$  and  $f(4) = 0$ ,  
then  $f(5) = 8$
- C2      If  $f(x)$  is a cubic function  
and  $f(1) = 0$ ,  $f(3) = 0$  and  $f(4) = 0$ ,  
then  $f(5)$  could be any number but zero.

# True or False?

- D1  $f(x) = (x - 2)^2(x - 7)$   
 $g(x) = (x - 2)^2(7 - x)$   
 $f(x)$  is a reflection of  $g(x)$  over the  $y$  axis
- D2  $f(x) = x(x - 2)^2$   
 $g(x) = -x(-x - 2)^2$   
 $f(x)$  is a reflection of  $g(x)$  over the  $y$  axis

# True or False?

E1       $f(x) = g(x + 2) \Rightarrow$   
 $g(x)$  is a horizontal translation of  $f(x)$

E2       $f(x) = g(x) + 2 \Rightarrow$   
 $g(x)$  is a horizontal translation of  $f(x)$

# Mathematics Assessment Project

## **Classroom Challenges**

These materials were designed and developed by the  
Shell Center Team at the Center for Research in Mathematical Education  
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**Malcolm Swan,**  
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with  
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who took part in the classroom trials that played a critical role in developing these materials

The classroom observation teams in the US were led by  
**David Foster, Mary Bouck, and Diane Schaefer**

This project was conceived and directed for  
The Mathematics Assessment Resource Service (MARS) by  
**Alan Schoenfeld** at the University of California, Berkeley, and  
**Hugh Burkhardt, Daniel Pead, and Malcolm Swan** at the University of Nottingham

Thanks also to Mat Crosier, Anne Floyde, Michael Galan, Judith Mills, Nick Orchard, and Alvaro Villanueva who contributed to the design and production of these materials

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We are particularly grateful to  
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The full collection of Mathematics Assessment Project materials is available from

<http://map.mathshell.org>