CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Classifying Rational and Irrational Numbers

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Classifying Rational and Irrational Numbers

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to distinguish between rational and irrational numbers. In particular, it aims to help you identify and assist students who have difficulties in:

- Classifying numbers as rational or irrational.
- Moving between different representations of rational and irrational numbers.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

N-RN: Use properties of rational and irrational numbers.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 3 and 6:

1. Make sense of problems and persevere in solving them.
3. Construct viable arguments and critique the reasoning of others.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

The lesson unit is structured in the following way:

- Before the lesson, students attempt the assessment task individually. You then review students' work and formulate questions that will help them improve their solutions.
- After a whole-class introduction, students work collaboratively in pairs or threes classifying numbers as rational and irrational, justifying and explaining their decisions to each other. Once complete, they compare and check their work with another group before a whole-class discussion, where they revisit some representations of numbers that could be either rational or irrational and compare their classification decisions.
- In a follow-up lesson, students work individually on a second assessment task.

MATERIALS REQUIRED

- Each individual student will need a mini-whiteboard, pen, and eraser, and a copy of *Is it Rational?* and *Classifying Rational and Irrational Numbers*.
- Each small group of students will need the *Poster Headings*, a copy of *Rational and Irrational Numbers (1) and (2)*, a large sheet of poster paper, scrap paper, and a glue stick.
- Have calculators and several copies of the *Hint Sheet* available in case students wish to use them.
- Either cut the resource sheets *Poster Headings*, *Rational and Irrational Numbers (1) and (2),* and *Hint Sheet* into cards before the lesson, or provide students with scissors to cut-up the cards themselves.
- You will need some large sticky notes and a marker pen for use in whole-class discussions.
- There is a projector resource to help with whole-class discussions.

TIME NEEDED

15 minutes before the lesson for the assessment task, a 1-hour lesson, and 20 minutes in a follow-up lesson. All timings are approximate and will depend on the needs of your students.

BEFORE THE LESSON

Assessment task: *Is it Rational?* (15 minutes)

Have students do this task in class or for homework a day or more before the formative assessment lesson. This will give you an opportunity to assess the work and identify students who have misconceptions or need other forms of help. You should then be able to target your help more effectively in the subsequent lesson.

Give each student a copy of *Is it Rational?*

I'd like you to work alone for this part of the lesson.

Spend 15 minutes answering these questions. Show all your work on the sheet and make sure you explain your answers really clearly.

I have some calculators if you wish to use one.

It is important that, as far as possible, students answer the questions without assistance. Help students to understand that they should not worry too much if they cannot understand or do everything because, in the next lesson, they will work on a related task that should help them make progress.

Is it Rational?

Remember that a bar over digits indicates a recurring decimal number, e.g. $0.2\overline{56} = 0.2565656...$

1. For each of the numbers below, decide whether it is rational or irrational.
   Explain your reasoning in detail.

5	
$\frac{5}{7}$	
0.575	
$\sqrt{5}$	
$5+\sqrt{7}$	
$\frac{\sqrt{10}}{2}$	
$5.75....$	
$(5+\sqrt{5})(5-\sqrt{5})$	
$(7+\sqrt{5})(5-\sqrt{5})$	

2. Arlo, Hao, Eiji, Korbin, and Hank were discussing $0.\overline{57}$.
   This is the script of their conversation.

Student	Statement	Agree or disagree?
Arlo:	$0.\overline{57}$ is an irrational number.	
Hao:	No, Arlo, it is rational, because $0.\overline{57}$ can be written as a fraction.	
Eiji:	Maybe Hao's correct, you know. Because $0.\overline{57} = \frac{57}{100}$.	
Korbin:	Hang on. The decimal for $0.\overline{57}$ would go on forever if you tried to write it. That's what the bar thing means, right?	
Hank:	And because it goes on forever, that *proves* $0.\overline{57}$ has *got* to be irrational.	

a. In the right hand column, write whether you agree or disagree with each student's statement.

b. If you think $0.\overline{57}$ is rational, say what fraction it is and explain why.

If you think it is not rational, explain how you know.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and any difficulties they encounter. The purpose of this is to forewarn you of the issues that will arise during the lesson, so that you may prepare carefully.

We suggest that you do not score students' work. The research shows that this is counterproductive, as it encourages students to compare scores and distracts their attention from how they may improve their mathematics.

Instead, help students to make progress by asking questions that focus attention on aspects of their work. Some suggestions for these are given in the *Common issues* table on pages T-3 and T-4. These have been drawn from common difficulties observed in trials of this unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students in the follow-up lesson.

Common issues:	**Suggested questions and prompts:**
Does not recognize rational numbers from simple representations For example: The student does not recognize integers as rational numbers. Or: The student does not recognize terminating decimals as rational numbers.	• A rational number can be written as a fraction of whole numbers. Is it possible to write 5 as a fraction using whole numbers? What about 0.575? • Are all fractions less than one?
Does not recognize non-terminating repeating decimals as rational For example: The student states that a non-terminating repeating decimal cannot be written as a fraction.	• Use a calculator to find $\frac{1}{9}, \frac{2}{9}, \frac{3}{9} \ldots$ as a decimal. • What fraction is $0.\bar{8}$? • What kind of decimal is $\frac{1}{3}$?
Does not recognize irrational numbers from simple representations For example: The student does not recognize $\sqrt{5}$ is irrational.	• Write the first few square numbers. Only these perfect square integers have whole number square roots. So which numbers can you find that have irrational square roots?
Assumes that all fractions are rational For example: The student claims $\frac{\sqrt{10}}{2}$ is rational.	• Are all fractions rational? • Show me a fraction that represents a rational/irrational number?
Does not simplify expressions involving radicals For example: The student assumes $(5+\sqrt{5})(5-\sqrt{5})$ is irrational because there is an irrational number in each parenthesis.	• What happens if you remove the parentheses? • Are all expressions that involve a radical irrational?
Explanations are poor For example: The student provides little or no reasoning.	• Suppose you were to explain this to someone unfamiliar with this type of work. How could you make this math clear, to help the student to understand?

Common issues:	Suggested questions and prompts:
Does not recognize that some representations are ambiguous For example: The student writes that 5.75... is rational or that it is irrational, not seeing that 5.75... is a truncated decimal that could continue in ways that represent rational numbers (such as $5.\overline{75}$), and that represent irrational numbers (non-terminating non-repeating decimals).	• The dots tell you that the digits would continue forever, but not how. Write a number that could continue but does repeat. And another… And another… • Now think about what kind of number this would be if subsequent digits were the same as the decimal expansion of π.
Does not recognize that repeating decimals are rational For example: The student agrees with Arlo that $0.\overline{57}$ is an irrational number. Or: The student disagrees with Hao, claiming $0.\overline{57}$ cannot be written as a fraction.	• How do you write $\frac{1}{3}$ as a decimal? What about $\frac{4}{9}$? • Does every rational number have a terminating decimal expansion?
Does not know how to convert repeating decimals to fraction form For example: The student makes an error when converting between representations (Q2b.)	• What is the difference between $0.\overline{57}$ and 0.57? • How do you write $\frac{1}{2}$ as a decimal? • How would you write $0.\overline{5}$ as a fraction? • Explain each stage of these calculations: • $x = 0.\overline{7}$, $10x = 7.\overline{7}$, $9x = 7$, $x = \frac{7}{9}$.
Does not interpret repeating decimal notation correctly For example: The student disagrees with Korbin, who said that the bar over the decimal digits means the decimal "would go on forever if you tried to write it out."	• Remember that a bar indicates that a decimal number is repeating. Write the first ten digits of these numbers: $0.\overline{45}$, $0.3\overline{45}$. Could you figure out the 100th digit in either number?
Does not understand that repeating non-terminating decimals are rational and non-repeating non-terminating decimals are irrational For example: The student agrees with Hank, that because $0.\overline{57}$ is non-terminating, it is irrational and does not distinguish non-repeating from repeating non-terminating decimals.	• How do you write $\frac{1}{3}$ as a decimal? What about $\frac{4}{9}$? • Does every rational number have a terminating decimal expansion? • Does every irrational number have a terminating decimal expansion? • Which non-terminating decimals can be written as fractions?

SUGGESTED LESSON OUTLINE

Introduction (10 minutes)

Give each student a mini-whiteboard, pen, and eraser. Use these to maximize participation in the introductory discussion.

Explain the structure of the lesson to students:

Recall your work on irrational and rational numbers [last lesson]. You'll have a chance to review that work in the lesson that follows today.

Today's lesson is to help you improve your solutions.

Display Slide P-1 of the projector resource:

Classifying Rational and Irrational Numbers

	Rational Numbers	Irrational Numbers
Terminating decimal		
Non-terminating repeating decimal		
Non-terminating non-repeating decimal		

Explain to students that this lesson they will make a poster classifying rational and irrational numbers.

I'm going to give you some headings and a large sheet of paper. You're going to use the headings to make this classification poster.

You will be given some cards with numbers on them and you have to classify them as rational or irrational, deciding where each number fits on your poster.

Check students' understanding of the terminology used for decimal numbers:

On your whiteboard, write a number with a terminating decimal.

Can you show a number with a non-terminating decimal on your whiteboard?

Show me a number with a repeating decimal.

Show me the first six digits of a non-repeating decimal.

Write $0.\overline{123}$ on a large sticky note. Model the classification activity using the number $0.\overline{123}$.

$0.\overline{123}$*. Remind me what the little bar over the digits means. [It is a repeating decimal that begins* $0.123123123...$ *; the digits continue in a repeating pattern; it does not terminate.]*

In which row of the table does $0.\overline{123}$ *go? Why? [Row 2, because the decimal does not terminate but does repeat.]*

Ok. So this number is a non-terminating repeating decimal because the bar shows it has endless repeats of the same three digits. [Write this on the card.]

Show me on your whiteboard: is $0.\overline{123}$ *rational or irrational? [Rational.]*

Students may offer different opinions on the rationality of $0.\overline{123}$. If there is dispute, accept students' answers for either classification at this stage in the lesson. Make it clear to students that the issue is unresolved and will be discussed again later in the lesson.

Collaborative small-group work (20 minutes)

Organize students into groups of two or three. For each group provide the *Poster Headings,* the sheets *Rational and Irrational Numbers (1) and (2)*, a large sheet of paper for making a poster, and scrap paper. Do not distribute the glue sticks yet.

Students take turns to place cards and collaborate on justifying these placements. Once they have agreed on a placement, the justification is written on the card and either placed on the poster or to one side.

Display Slide P-2 of the projector resource and explain to students how you expect them to collaborate:

Instructions for Placing Number Cards

- Take turns to choose a number card.
- When it is your turn:
  - Decide where your number card fits on the poster.
  - Does it fit in just one place, or in more than one place?
  - Give reasons for your decisions.
- When it is your partner's turn:
  - If you agree with your partner's reasoning, explain it in your own words.
  - If you disagree with your partner's decision, explain why. Then together, figure out where to put the card.
- When you have reached an agreement:
  - Write reasons for your decision on the number card.
  - If the number card fits in just one place on the poster, place it on the poster.
  - If not, put it to one side.

Here are some instructions for working together.

All students in your group should be able to give reasons for every placement.

Don't glue things in place or draw in the lines on the poster yet, as you may change your mind later.

During small group work you have two tasks: to find out about students' work and to support their thinking.

Find out about students' work

Listen carefully to students' conversations. Note especially difficulties that emerge for more than one group.

Do students assume that all fractions represent rational numbers? Do students manipulate the expressions involving radicals to show that the number represented is rational/irrational? Or do they evaluate the expressions on a calculator? Figure out which students can convert a non-terminating repeating decimal to a fraction. Do students identify all the meanings of ambiguous expressions such as '0.123 rounded correct to 3 decimal places'?

Listen for the kind of reasoning students give in support of their classifications. Do they use definitions? Do they reason using analogies with other examples?

You can use what you hear to identify students to call on in the whole-class discussion and in particular, find two or three cards as a focus for that discussion.

Support student thinking

If you hear students providing incorrect classifications or justifications, try not to resolve the issues for them directly. Instead, ask questions that help them to identify their errors and to redirect their thinking. You may want to use some of the questions and prompts from the *Common issues* table. If students are relying on a calculator to place the cards, encourage them to explain the answers displayed on it.

If a student struggles to get started, encourage them to ask a specific question about the task. Articulating the problem in this way can sometimes offer ideas to pursue that were previously overlooked. However, if the student needs their question answered, ask another member of the group for a response. There is a *Hint Sheet* provided to help students who struggle to make progress in classifying these numbers: $0.1\overline{23}$, $\frac{\sqrt{3}}{4}$, the calculator display 3.14159265, and $0.\bar{9}$.

If one group of students has difficulty with a problem another group dealt with well, you could ask them to talk to each other. If several groups of students are finding the same issue difficult, you might write a suitable question on the board, or organize a brief whole-class discussion focusing on that aspect of mathematics.

Are all fractions rational? Show me a fraction that is rational/that is irrational.

What does a calculator display 0.7777777778 / 0.1457878342 tell you about the number?

Is any number with a root sign irrational?

Prompt students to write reasons for their decisions next to the cards. If you hear one student providing a justification, prompt the other members of the group to either challenge or rephrase what they heard.

If any groups finish early, ask them to use the blank card to try to make up a new number to fit in an empty cell on the poster.

A couple of minutes before the end of the activity, ask each group to write onto a sheet of scrap paper the numbers from cards they have decided not to place on the poster.

Comparing solutions (10 minutes)

Ask one student from each group to swap with a student in another group, taking with them the sheet of scrap paper on which they have written their group's non-classified numbers.

In the new groups, students compare the numbers they have not classified, to see if there are any differences. Ask students to share their reasons for the numbers they have not classified.

Gluing posters (5 minutes)

Ask students to return to their original small groups and distribute glue sticks. Ask them to discuss with their partners any changes they might want to make. Once students are satisfied with their answers, they can glue the cards in place. Remind students **not** to put on the poster any number cards they think can go in more than one place.

While students work on this, think about the numbers your students found difficult to place, or numbers for which you know there are different solutions. You can use these numbers as a focus for the whole-class discussion. Write these numbers in marker pen on large sticky notes.

Whole-class discussion (15 minutes)

In this discussion, we suggest you focus on reasoning about one or two examples that students found difficult, rather than checking students all have the same classifications.

Check that each student has a mini-whiteboard, pen, and eraser. Display Slide P-1 *Classifying Rational and Irrational Numbers* again. Use the numbers you wrote on sticky notes: you will be able to move them from one position to another on the classification table until students have reached agreement about proper placement.

Choose one of your sticky notes and read it to the class. Ask students to write on their mini-whiteboards where to place it on the classification table. Place your sticky note on the part of the table indicated by one student and ask them to justify their placement of the card. Now ask another student to comment on the justification. Call on students who gave different answers on their whiteboards.

Bradley, where did you place this card? What was your reasoning?

Do you agree, Lia? You do? Please explain Bradley's answer in your own words.

Kanya, can you tell us why you think differently?

Move the cards from one cell to another until students reach agreement on the placement of the card. If there is no disagreement, provoke some by placing a number incorrectly and asking students for an explanation.

The discussion below is indicative. You may feel that other issues should be the focus of discussion for your students.

Sample discussion

You might begin by checking the placement of the non-terminating repeating decimal $0.\overline{123}$.

Write on your mini-whiteboard. Is $0.\overline{123}$ *terminating or non-terminating?*

Is $0.\overline{123}$ *rational or irrational?*

Alicia, where did you place $0.\overline{123}$*? Why might you think it goes there?*

As part of this discussion, you might ask students to show on their mini-whiteboards how to change from the repeating non-terminating decimal representation to a fraction. If necessary, ask students to work with you, as you model use of the standard algorithm for changing a repeating decimal to a fraction.

Focus on the decimal representations 0.123, $0.\overline{123}$, $0.123...$, $0.1\overline{23}$, and 0.123 (rounded correct to three decimal places). To actively involve students, ask them to use their mini-whiteboards.

Are $0.\overline{123}$ *and* 0.123 *the same number? How do you know?*

Is $0.\overline{123}$ *rational or irrational? How do you know?*

What about 0.123*? Is that rational or irrational? How do you know?*

What is the difference between 0.123 *and* $0.123...$*? What difference do those little dots [the ellipsis] make?*

Show me a way $0.123...$ *might continue if it's a rational number.*

Show me a way $0.123...$ *might continue that would make it an irrational number. Why is this difficult?*

Push students to identify explicitly the difference between decimals in which there is a repeating pattern of digits and non-terminating decimals in which there is a pattern, but no repeats.

Either refer to one of the student's whiteboards, or write $0.123456789101112131415...$ on the board.

What would the next six digits be, Jenny? How do you know?
Tell me about the pattern in the digits of this decimal number.
Is this a repeating decimal?
Do the same digits repeat in the same order again and again?

Consider the other ambiguous representations:

What if the number is rounded to 0.123*? Is this a rational or an irrational number?*
Show me a number that could be rounded to 0.123 *that is a rational number.*
Show me a number that could be rounded to 0.123 *that is an irrational number.*

Did you find any other representations that might be rational, or might be irrational, depending on how the number continued? [The calculator display looks like the first digits of pi, but could be a rational number.]

Explain how the calculator display number might continue.
Show me a rational version of the calculator display number.

If there is time, consider $0.\bar{9}$ and how, as this can be written as 1, the distinction between terminating and non-terminating decimal representations becomes blurred.

Finally, discuss the empty cells on the classification table (irrational terminating decimal, irrational non-terminating repeating decimal, rational non-terminating non-repeating decimal).

Which numbers go in this cell?

Is it possible to place numbers in this cell? [No. No terminating decimals/non-terminating repeating decimals are irrational numbers and no non-terminating non-repeating decimals are rational numbers.]

Follow-up lesson: individual review (20 minutes)

Give each student his or her original solution to the assessment task, *Is it Rational?* along with a new copy of the same task. If you have not added questions to individual pieces of work, write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

Invite students to revise their work using your questions:

I would like you to read through your original solutions and the questions I have written (on the board/on your script) to help you improve your work.

Answer these questions and revise your response.

Now give each student a copy of the *Classifying Rational and Irrational Numbers* assessment task.

Use what you have learned to answer these questions. Show all your work on the sheet and make sure you explain your answers really clearly. I have some calculators if you wish to use one.

If students struggled with the original assessment task, you may feel it more appropriate for them to just spend their time revisiting *Is it Rational?* rather than attempting *Classifying Rational and Irrational Numbers* as well.

Some teachers give this as a homework task.

SOLUTIONS

Assessment task: *Is it Rational?*

1.

Number	Possible student reasoning
5	5 is rational. It can be written as a fraction, the ratio of two integers *a, b,* with $b \neq 0$, e.g. $\frac{35}{7}, \frac{-5}{-1}$. Students sometimes discount whole number rationals, especially 0.
$\frac{5}{7}$	$\frac{5}{7}$ is rational, since it can be written as the ratio of two integers *a, b,* with $b \neq 0$.
0.575	0.575 is rational. Using decimal place value, $0.575 = \frac{575}{1000}$.
$\sqrt{5}$	$\sqrt{5}$ is irrational. Students might argue that if a number is not a perfect square then its root is irrational. Or they might argue that $\sqrt{4} < \sqrt{5} < \sqrt{9}$, and there is no integer between 2 and 3. This doesn't establish that there is no fractional value for $\sqrt{5}$ though. At this stage of their math learning, students are unlikely to produce a standard proof by contradiction of the irrationality of $\sqrt{5}$.
$5+\sqrt{7}$	$5+\sqrt{7}$ is irrational because the sum of a rational and irrational number is always irrational.
$\frac{\sqrt{10}}{2}$	$\frac{\sqrt{10}}{2}$ is irrational. Students may think it is rational because it is represented as a fraction.
5.75....	5.75.... From this truncated decimal representation, it is not possible to decide whether the number represented by 5.75.... is rational or irrational. This is a difficulty that students often encounter when reading calculator displays. The dots here indicate that the decimal is non-terminating, but the number might have a repeating (rational) or non-repeating (irrational) tail. Students could give examples to show this, e.g. $5.75\bar{7}$ and 5.75123456789101112131415.…
$(5+\sqrt{5})(5-\sqrt{5})$	The product of these two irrational factors is rational. Using a calculator will suggest that the product is rational, but not why. $(5+\sqrt{5})(5-\sqrt{5}) = 25 - 5\sqrt{5} + 5\sqrt{5} - 5 = 20$. Removing the parentheses and simplifying does provide a reason why the expression is rational.
$(7+\sqrt{5})(5-\sqrt{5})$	The product of these two irrational factors is irrational. $(7+\sqrt{5})(5-\sqrt{5}) = 35 - 7\sqrt{5} + 5\sqrt{5} - 5 = 30 - 2\sqrt{5}$. Since $\sqrt{5}$ is irrational, $2\sqrt{5}$ is irrational and $30-2\sqrt{5}$ is irrational. Use of a calculator will not help to show the irrationality of $30-2\sqrt{5}$.

2.

Student	Statement	Agree or disagree?	
Arlo:	$0.\overline{57}$ is an irrational number.	Disagree	Agreeing with Arlo, that $0.\overline{57}$ is an irrational number, may indicate a standard misconception. Many students believe that a number with an infinitely long decimal representation must be irrational.
Hao:	No, Arlo, it is rational, because $0.\overline{57}$ can be written as a fraction.	Agree	Hao is correct. $0.\overline{57} = \frac{19}{33}$
Eiji:	Maybe Hao's correct, you know. Because $0.\overline{57} = \frac{57}{100}$.	Agree with statement, but disagree with reason.	Eiji is correct in that Hao's answer is correct, but he is incorrect to say that $0.\overline{57} = \frac{57}{100}$. This is a common error.
Korbin:	Hang on. The decimal for $0.\overline{57}$ would go on forever if you tried to write it out. That's what the bar thing means, right?	Agree	$0.\overline{57} = 0.57575757...$ does "go on forever." Not recognizing this is a misunderstanding of standard notation.
Hank:	And because it goes on forever, that proves $0.\overline{57}$ has got to be irrational.	Disagree	Thinking that an infinitely long decimal tail means a number can't be rational is a common student error.

2. b. $0.\overline{57}$ is rational.

Let $x = 0.\overline{57}$. Then $100x = 57.\overline{57}$. So $99x = 57$. Thus $x = \frac{57}{99} = \frac{19}{33}$.

Collaborative small-group work

	Rational numbers	Irrational numbers
Terminating decimal	$\frac{7}{8} = 0.875$ and is a terminating decimal.	This cell is empty: there are no terminating decimal representations of irrational numbers.
	$0.123 = \frac{123}{1000}$ and so rational and terminating.	
	$(8+\sqrt{2})(8-\sqrt{2}) = 64 - 8\sqrt{2} + 8\sqrt{2} - \sqrt{2}^2 = 64 - 2 = 62.$ 62 rational, with a terminating decimal representation.	
	$\frac{\sqrt{8}}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}} = 2$ 2 is rational, with a terminating decimal representation.	
	$\sqrt{2} \times \sqrt{8} = \sqrt{2} \times 2\sqrt{2} = 2 \times 2 = 4$ 4 is rational, with a terminating decimal representation.	
Non-terminating repeating decimal	$\frac{2}{3} = 0.\overline{6}$ Students might already know this decimal conversion, or perform division to calculate it.	This cell is empty: there are no non-terminating repeating decimal representations of irrational numbers.
	$\frac{22}{7} = 3.\overline{142897}$ Students might use division to calculate this commonly used approximation of π. Using a calculator, they may not find all the repeating digits and so would have no reason to suppose it is a repeating decimal.	
	$x = 0.\overline{123}$ $1000x = 123.\overline{123}$ $1000x - x = 123$ $999x = 123$ $x = 0.\overline{123} = \frac{123}{999}$	
	$x = 0.1\overline{23}$ $10x = 1.\overline{23}$ $1000x = 123.\overline{23}$ $1000x - 10x = 122$ $x = \frac{122}{990} = \frac{61}{495}$	

	Rational numbers	Irrational numbers
Non-terminating non-repeating decimal	This cell is empty: no rational number has a non-terminating non-repeating decimal representation.	π is irrational, with a non-terminating, non-repeating decimal representation. We expect students will have been told of the irrationality of π and do not expect this claim to be justified.
		$\frac{\sqrt{3}}{4}$ is irrational since 3 is not a perfect square and only integers that are perfect squares have rational roots. An irrational divided by a rational is irrational.
		$\sqrt{8}$ is irrational since 8 is not a square number and only integers that are square numbers have rational roots.
		$\sqrt{2}+\sqrt{8}$ is the sum of two irrational numbers. It is irrational, since $\sqrt{2}+\sqrt{8}=\sqrt{2}+2\sqrt{2}=3\sqrt{2}$ and the product of a non-zero rational number and an irrational number is irrational. Some sums of irrational numbers are rational, such as $4\sqrt{2}-2\sqrt{8}$.

There is insufficient information to assign the following number cards to a unique category:

0.123 ....

This is a non-terminating decimal, but the ellipsis does not determine whether it is repeating or non-repeating.

It could turn out to be a rational number, such as $0.123\overline{4}$.

It could turn out to be an irrational number, such as 0.12301001000100001… (a patterned tail of zeros and ones, with one extra zero between ones each time.)

0.123 (rounded correct to three decimal places) and the calculator display: 3.14159265

0.123 could be a rounded version of either a rational or an irrational number, because it might be a rounded terminating decimal, rounded non-terminating repeating decimal (both rational), or a non-terminating non-repeating decimal (irrational). This is also the case with the calculator display, which shows the first 9 digits of pi.

$0.\overline{9}$ is a separate case.

It is clearly a rational number, but its categorization as a non-terminating repeating decimal, rather than as a terminating decimal is dubious.

Students might argue that $0.\overline{1}=\frac{1}{9}$, $0.\overline{2}=\frac{2}{9}$, … , $0.\overline{9}=\frac{9}{9}=1$.

Alternatively, they could show that if $x=0.\overline{9}$, $10x=9.\overline{9}$ so $9x=9$ and $x=\frac{9}{9}=1$.

A similar argument could be made for any of the numbers with terminating decimal expansions.

Assessment task: *Classifying Rational and Irrational Numbers*

1.

Number	Possible student reasoning
0.21	0.21 is a rational number. A student might establish this by showing that is can be written as a fraction of integers e.g. $\frac{21}{100}$ or by stating that it is a terminating decimal, and therefore rational.
$\frac{3}{12}$	$\frac{3}{12}$ is the ratio of two integers *a, b,* with $b \neq 0$ and so is rational. Or alternatively, $\frac{3}{12} = 0.25$, which is a terminating decimal and so is rational.
$\sqrt{12} - 2$	Students may argue that $\sqrt{12}$ is irrational because $\sqrt{12} = \sqrt{3 \times 4} = 2 \times \sqrt{3}$ and the product of a non-zero rational and an irrational number is irrational.
$\frac{\sqrt{12}}{4}$	$\frac{\sqrt{12}}{4} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$. The quotient of an irrational and a non-zero integer is irrational.
4.125...	The ellipsis does not determine how the decimal digits will continue, only that they do. The number could be irrational (the digits continue and do not repeat) or rational (the digits continue and repeat). Check whether students confuse "non-repeating" with "non-patterned": a number in which the pattern of the digits is e.g. 4.12512612712812912101211… is irrational because although patterned, the non-terminating tail will never repeat itself.
$(\sqrt{12} - 4)(4 + \sqrt{12})$	$(\sqrt{12} - 4)(4 + \sqrt{12}) = 4\sqrt{12} + 12 - 16 - 4\sqrt{12} = -4$. This is a rational number.
12.52 (rounded to 2 d.p.)	There is no way of telling whether this is rational or irrational. For example: 12.521234567891011…. is irrational (providing the pattern continues) but 12.524 is rational.

2. In this question, the student's explanations are more important than the truth value assigned to the statement. These explanations may indicate the student still makes common errors or has alternative conceptions.

Student	Statement	Agree or disagree?	
Otis	$\frac{\sqrt{3}}{8}$ is a rational number because it can be written as a fraction.	Disagree	Otis is incorrect. Students who agree with Otis may not understand that some fractions are irrational! That error comes from a partial definition of rational. A fuller definition of rational number is that it is a number that can be written as a ratio of integers.
Lulu	$\frac{\sqrt{3}}{8}$ is irrational because $\sqrt{3}$ is irrational.	Agree	Lulu's statement is correct, but her justification is incomplete. $\frac{\sqrt{3}}{8}$ is irrational because $\sqrt{3}$ is irrational and the quotient of an irrational number and non-zero integer is irrational.
Leon	$0.\overline{286}$ is rational because you can write it as the fraction $\frac{286}{1000}$.	Agree with statement, but disagree with reason.	Leon is correct that $0.\overline{286}$ is rational, but his claim that it is the fraction $\frac{286}{1000}$ is incorrect. Students may produce the correct rational representation of the recurring decimal: $0.\overline{286} = \frac{286}{999}$. Alternatively, students may argue that $\frac{286}{1000} = 0.286 \neq 0.\overline{286}$.
Joan	$0.\overline{286}$ is an irrational number because that decimal will carry on forever.	Disagree	Joan is incorrect to claim that $0.\overline{286}$ is irrational. A repeating non-terminating decimal is a rational number.
Ray	0.286 (rounded to three decimal places) might be rational or irrational.	Agree	Ray is correct. If 0.286 were rounded from e.g. 0.2855555555… then it would be a rational number. If it were rounded from 0.2861234567891011121314… then, since this pattern does not repeat, the number would be irrational.
Arita	0.286… is rational - the little dots show the digits carry on in the same pattern forever.	Disagree	Arita is incorrect. The ellipsis shows only that the digits do continue, not how. 0.286... could represent either a rational or an irrational number

Is it Rational?

Remember that a bar over digits indicates a recurring decimal number, e.g. $0.2\overline{56} = 0.2565656...$

1. For each of the numbers below, decide whether it is rational or irrational.

   Explain your reasoning in detail.

5	
$\frac{5}{7}$	
0.575	
$\sqrt{5}$	
$5+\sqrt{7}$	
$\frac{\sqrt{10}}{2}$	
$5.75....$	
$(5+\sqrt{5})(5-\sqrt{5})$	
$(7+\sqrt{5})(5-\sqrt{5})$	

2. Arlo, Hao, Eiji, Korbin, and Hank were discussing $0.\overline{57}$.

This is the script of their conversation.

Student	**Statement**	**Agree or disagree?**
Arlo:	$0.\overline{57}$ is an irrational number.	
Hao:	No, Arlo, it is rational, because $0.\overline{57}$ can be written as a fraction.	
Eiji:	Maybe Hao's correct, you know. Because $0.\overline{57} = \frac{57}{100}$.	
Korbin:	Hang on. The decimal for $0.\overline{57}$ would go on forever if you tried to write it. That's what the bar thing means, right?	
Hank:	And because it goes on forever, that *proves* $0.\overline{57}$ has *got* to be irrational.	

a. In the right hand column, write whether you agree or disagree with each student's statement.

b. If you think $0.\overline{57}$ is rational, say what fraction it is and explain why.

If you think it is not rational, explain how you know.

Poster Headings

Classifying Rational and Irrational Numbers	Terminating decimal	Rational numbers	
	Non-terminating non-repeating decimal	Non-terminating repeating decimal	Irrational numbers

Rational and Irrational Numbers (1)

$\frac{7}{8}$	$0.123\ldots$	$0.\bar{9}$
$\frac{2}{3}$	$\frac{22}{7}$	$(8+\sqrt{2})(8-\sqrt{2})$
π	$\frac{\sqrt{8}}{\sqrt{2}}$	$\sqrt{8}$

Rational and Irrational Numbers (2)

$\sqrt{2} \times \sqrt{8}$	$\sqrt{2} + \sqrt{8}$	$\dfrac{\sqrt{3}}{4}$
$0.\overline{123}$	0.123 rounded correct to 3 decimal places	Calculator display: 3.14159265
0.123	$0.1\overline{23}$	

Hint Sheet

$0.\overline{123}$ Figure out the changes from one line to the next: $x = 0.\overline{43}$ $100x = 43.\overline{43}$ $99x = 43$ $x = \frac{43}{99}$ How can this help you figure out $0.\overline{123}$ as a fraction?	$\frac{\sqrt{3}}{4}$ What kind of number is $\sqrt{3}$? What is your definition of a rational number? Are all fractions rational numbers?
Calculator display: `3.14159265` Does this calculator display show π? Why might you say *yes*? Why might you say *no*?	$0.\overline{9}$ Write $\frac{1}{9}$ as a decimal. Write $\frac{2}{9}$ as a decimal. What fraction would you consider next? Continue the pattern.

Classifying Rational and Irrational Numbers

1. For each of the numbers below, decide whether it is rational or irrational.

   Explain your answers.

Number	Reasoning
0.21	
$\frac{3}{12}$	
$\sqrt{12}-2$	
$\frac{\sqrt{12}}{4}$	
$4.125...$	
$(\sqrt{12}-4)(4+\sqrt{12})$	
12.52 (rounded to 2 d.p.)	

2. Some students were classifying numbers as rational and irrational.

Decide whether you agree or disagree with each statement.

Correct any errors. Explain your answers clearly.

Student	Statement	Agree or disagree?
Otis	$\frac{\sqrt{3}}{8}$ is a rational number because it can be written as a fraction.	
Lulu	$\frac{\sqrt{3}}{8}$ is irrational because $\sqrt{3}$ is irrational.	
Leon	$0.\overline{286}$ is rational because you can write it as the fraction $\frac{286}{1000}$.	
Joan	$0.\overline{286}$ is an irrational number because that decimal will carry on forever.	
Ray	0.286 (rounded to three decimal places) might be rational or irrational.	
Arita	0.286… is rational - the little dots show the digits carry on in the same pattern forever.	

Classifying Rational and Irrational Numbers

	Rational Numbers	Irrational Numbers
Terminating decimal		
Non-terminating repeating decimal		
Non-terminating non-repeating decimal		

Instructions for Placing Number Cards

- Take turns to choose a number card.
- When it is your turn:
  - Decide where your number card fits on the poster.
  - Does it fit in just one place, or in more than one place?
  - Give reasons for your decisions.
- When it is your partner's turn:
  - If you agree with your partner's reasoning, explain it in your own words.
  - If you disagree with your partner's decision, explain why. Then together, figure out where to put the card.
- When you have reached an agreement:
  - Write reasons for your decision on the number card.
  - If the number card fits in just one place on the poster, place it on the poster.
  - If not, put it to one side.

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
Shell Center Team at the Center for Research in Mathematical Education
University of Nottingham, England:

Malcolm Swan,
Nichola Clarke, Clare Dawson, Sheila Evans, Colin Foster, and **Marie Joubert**
with
Hugh Burkhardt, Rita Crust, Andy Noyes, and **Daniel Pead**

We are grateful to the many teachers and students, in the UK and the US,
who took part in the classroom trials that played a critical role in developing these materials

The classroom observation teams in the US were led by
David Foster, Mary Bouck, and **Diane Schaefer**

This project was conceived and directed for
The Mathematics Assessment Resource Service (MARS) by
Alan Schoenfeld at the University of California, Berkeley, and
Hugh Burkhardt, Daniel Pead, and **Malcolm Swan** at the University of Nottingham

Thanks also to Mat Crosier, Anne Floyde, Michael Galan, Judith Mills, Nick Orchard, and Alvaro Villanueva who contributed to the design and production of these materials

This development would not have been possible without the support of
Bill & Melinda Gates Foundation

We are particularly grateful to
Carina Wong, Melissa Chabran, and **Jamie McKee**

The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org