CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Identifying Similar Triangles

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Identifying Similar Triangles

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how students reason about geometry and, in particular, how well they are able to:

- Use facts about the angle sum and exterior angles of triangles to calculate missing angles.
- Apply angle theorems to parallel lines cut by a transversal.
- Interpret geometrical diagrams using mathematical properties to identify similarity of triangles.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

8G: Understand congruence and similarity using physical models, transparencies, or geometry software.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with particular emphasis on Practices 3 and 6:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.

INTRODUCTION

- Before the lesson, students work individually on an assessment task designed to reveal their current understanding and ability to reason using geometrical properties. You then review their responses and create questions for students to consider when improving their work.
- After a whole-class introduction, students work in small groups on a collaborative discussion task, categorizing diagrams of pairs of triangles based on their similarity. Throughout their work, students justify and explain their thinking and reasoning.
- Students review their work by comparing their categorizations with those of their peers.
- In a whole-class discussion, students review their work and discuss what they have learned.
- In a follow-up lesson, students review their initial work on the assessment task and work alone on a task similar to the introductory task.

MATERIALS REQUIRED

- Each student will need a mini-whiteboard, pen, and wipe, and copies of the assessment tasks *Puzzling Triangles* and *Puzzling Triangles (revisited)*.
- Each small group of students will need a copy of *Sorting Triangles*, a pencil, a marker, a large sheet of poster paper, a pair of scissors and a glue stick.

There are also some *Blank Cards* for students to create some new cards for others to work on if they finish the collaborative lesson activity.

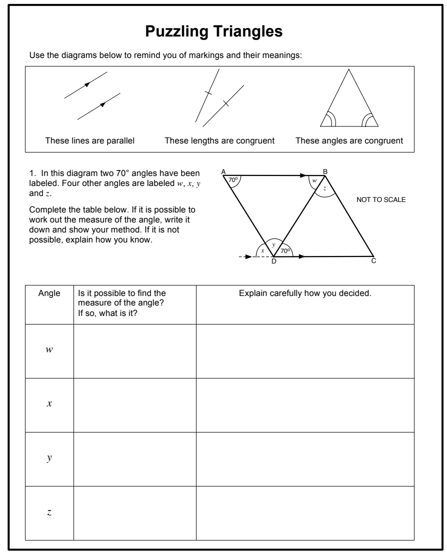
TIME NEEDED

15 minutes before the lesson, a 80-minute lesson (or split into two shorter lessons), and 15 minutes in a follow-up lesson (or for homework). These timings are not exact.

BEFORE THE LESSON

Assessment task: *Puzzling Triangles* (15 minutes)

Ask students to complete this task in class or for homework a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the next lesson.


Puzzling Triangles

Use the diagrams below to remind you of markings and their meanings:

1. In this diagram two 70° angles have been labeled. Four other angles are labeled w, x, y and z.

Complete the table below. If it is possible to work out the measure of the angle, write it down and show your method. If it is not possible, explain how you know.

Angle	Is it possible to find the measure of the angle? If so, what is it?	Explain carefully how you decided.
w		
x		
y		
z		

Give each student a copy of the assessment task *Puzzling Triangles* and briefly introduce the task:

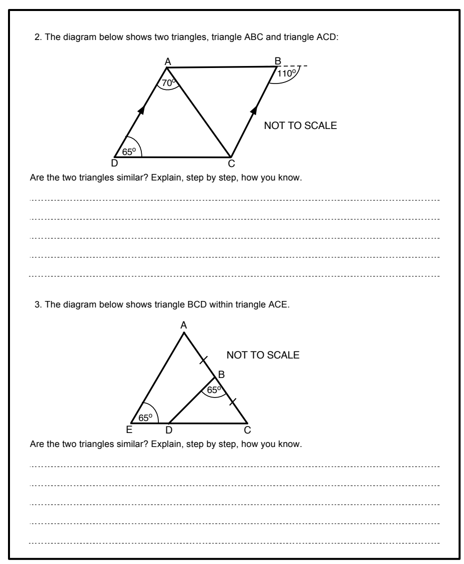
> *In this task you are asked whether it is possible to find the measure of some missing angles and to decide whether or not pairs of triangles are mathematically similar.*
>
> *There is an information box at the start of the task. Use this to help you to interpret the diagrams.*
>
> *Make sure you explain your answers clearly.*

It is important that, as far as possible, students are allowed to answer the questions without assistance.

Students should not worry too much if they cannot understand or do everything because in the next lesson they will work on a similar task that should help them. Explain to students that by the end of the next lesson they should be able to answer questions such as these confidently. This is their goal.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding, and their different ways of reasoning. The purpose of doing this is to forewarn you of issues that may arise during the lesson itself, so that you can prepare carefully. We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics. Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table that follows. These have been drawn from common difficulties observed in trials of this lesson unit.


2. The diagram below shows two triangles, triangle ABC and triangle ACD:

Are the two triangles similar? Explain, step by step, how you know.

3. The diagram below shows triangle BCD within triangle ACE.

Are the two triangles similar? Explain, step by step, how you know.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- Write one or two questions on each student's work, or
- Give each student a printed version of your list of questions, and highlight appropriate questions for each student.

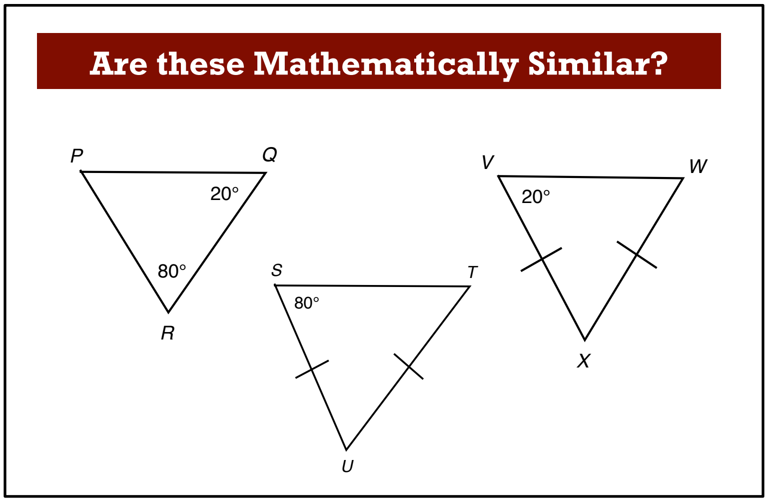
If you do not have time to do this, you could select a few questions that will be of help to the majority of students, and write these on the board when you return the work to the students

Common issues:	Suggested questions and prompts:
Student has difficulty getting started	• What properties of parallel lines do you know? • What triangle properties do you know? • Can you use these properties on the diagrams given?
Student relies on visual appearance For example: The student measures the missing angles. Or: The student says angles are/are not equal because they look/do not look the same. Or: The student assumes that lines that are not marked as parallel are parallel because they look a constant distance apart.	• The diagram has not been drawn to scale. Can you use mathematics to calculate the missing angle/show that the angles are the same? • These lines look like they might be parallel but are not marked as parallel. How can we check that they are or are not parallel? Which angles would be the same if they were parallel?
Student does not provide reasons for assertions For example: The student provides numerical answers with no justification. Or: The student makes a claim of equality without justification.	• Suppose someone did not believe your answer. How could you convince them you are correct? • Why do you think that (these angles are equal)? Explain how you know your answer is correct. • What properties are written on the diagram? Can you explain how you have used these properties?
Student provides incorrect reasons for assertions For example: The student refers to a theorem that does not apply to the given situation.	• Look at the diagram carefully. What features would the diagram need to have for this theorem to apply? • What other properties/theorems do you know that might apply to this diagram? How could you check? • Describe this situation using math properties. What else do you know about these properties?
Student produces only short chains of reasoning For example: The student can derive one claim, but does not combine derivations to produce a full solution.	• Write what you have found so far in detail. Do you see any connections? • What do you already know? What do you want to find out?
Student relies on just one form of reasoning For example: The student attempts to answer all questions using angle theorems.	• Are there any other properties that can be used in these diagrams? • What do you know about the angles in a triangle/on a straight line?

SUGGESTED LESSON OUTLINE

Whole-class introduction (15 minutes)

Give each student a mini-whiteboard, pen, and wipe. Display Slide P-1 of the projector resource:



Are these triangles mathematically similar?

Some students may assume that the triangles look different so none are similar.

None of the diagrams that you will be looking at today are drawn to scale so we cannot rely on what we can see or measure. In today's lesson you are going to be using mathematical reasoning to identify mathematically similar triangles.

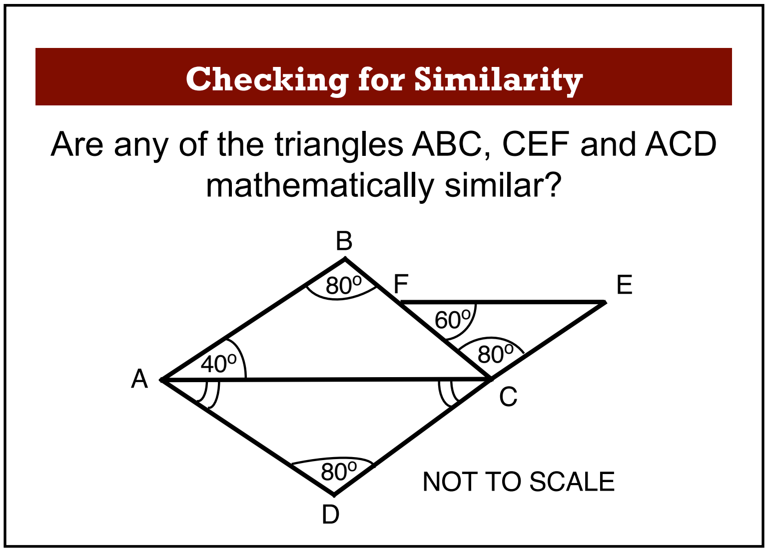
Write on your mini whiteboards what you think the words 'mathematically similar' mean. [They have the same shape but do not need to be the same size, or one may be transformed into the other by a sequence of rotations, reflections, translations, and dilations.]

Ask students to show you what they have written and follow up any misconceptions.

Now try to answer the question on the slide and write your reasoning on your mini whiteboards.

Encourage students to justify their reasoning carefully, using only the information on the diagrams. They should reason that triangles PQR and STU are similar because if they were drawn correctly, they would be the same shape (both with angles 80°, 80°, 20°) but not necessarily the same size. Triangle VWX will have angles 20°, 20°, 140° and so will be a completely different shape. Triangle VWX has been very badly drawn!

Now display Slide P-2 of the projector resource:



We have been given information about their angles but not about their lengths.
How do we show that two triangles are similar based on their angles alone?
[If two angles of one triangle are congruent to two angles of another triangle, then they are similar.]

Work with the class to fill in any missing angles. Encourage them to use their mini-whiteboards as they work. Check that they understand the markings showing congruent angles in the bottom triangle and that they know this is an isosceles triangle. Students should be able to conclude that triangles ABC and CEF are similar. Watch for students that assume that DCE is a straight line and reach an incorrect conclusion or a contradiction.

It may be appropriate to extend the discussion further by asking students to consider what extra information can be found out about the diagram by filling in the missing angles.

Which lines are parallel? [AC and FE are parallel, so are AB and CE.]
How do you know? [The alternate interior angles are 60° in the first case and 80° in the second.]

Is DCE a straight line? [No.]
How can you tell? [The three internal angles add to 190°: 80°+60°+50°.]

What markings can be added to triangle ACD?
[The two congruent angles imply an isosceles triangle, so two congruent sides may be marked.]

Revise previous knowledge about parallel lines and transversals if necessary.

Collaborative small group work (30 minutes)

Ask students to work in groups of two or three. Give each group a copy of *Sorting Triangles*, a pencil, a pair of scissors, a glue stick, a marker and a large sheet of poster paper.

Today you will be looking at pairs of triangles and deciding whether the triangles are 'similar', 'not similar', or it 'cannot be determined'.

Divide your large sheet of poster paper into three columns and head separate columns with the words: 'Similar', 'Not Similar', 'Cannot be Determined'.

For two triangles to be 'similar', what do you need to show?
[One way is to show that two angles of one triangle are congruent to two angles of the other.]

For two triangles to be 'not similar' what do you need to show?
[That the triangles contain at most one congruent angle.]

To show that it 'cannot be determined' you will need to show that we do not have enough information to show that the triangles have two congruent angles.

Use slide P-3 to explain how students are to work together:

Working Together

Take turns to:

1. Select a diagram, and decide whether or not the two triangles are similar, not similar, or 'it cannot be determined'.
   Explain your reasoning.
2. Fill in as many angles as you can.
3. Your partner must challenge your explanation if they disagree or describe it in their own words if they agree.
4. Once agreed, glue the diagram onto the poster and write your explanation **in pencil** next to it.
5. Continue to take turns until all the diagrams are sorted.

The purpose of this structured work is to encourage each student to engage with their partner's explanations and to take responsibility for their partner's understanding. Students should use their mini-whiteboards for calculations and to explain their thinking to each other.

Most of the triangles have angles labeled but some do not, so students will need to use angle theorems to label angles as congruent rather than calculating missing angles for these diagrams. If you think students will struggle with this, encourage them to begin with diagrams A and B.

It does not matter if students do not manage to place all the diagrams. It is more important that everyone in each group understands the categorization of each diagram.

While students are working, you have two tasks: to notice their approaches to the task, and to support student problem solving.

Notice different student approaches to the task

Listen to and watch students carefully. In particular, notice whether students are addressing the difficulties they experienced in the assessment task. Are students engaging with mathematical properties or are they relying on perceptual reasoning about surface features? Do students fully understand the notations they are working with? Are they noting the properties they read from diagrams? Do they notice any implications or connections of those properties?

Do students reason by deriving new information from known facts? Do they use theorems about the sum of angles on a straight line, the sum of angles in a triangle, angles formed when a transversal crosses a pair of parallel lines?

How do students refer to diagrams and properties? Do they use 'this' and 'that' frequently, or name the objects and properties? Do partners understand what the student is pointing to? If this is proving a problem, encourage students to use the labels on the diagrams and then explain with their hands behind their backs!

Support student problem solving

Help students to work constructively together. Remind them to look at the slide for instructions on how to take turns. Check that one student listens to another by asking the listener to paraphrase the speaker's statements. Check that students are recording their discussions as rough notes on their mini-whiteboards.

Try not to solve students' problems for them or to add structure to longer problems and do the reasoning for them. Instead, you might ask strategic questions to suggest ways of moving forward.

What do you know? What else can you find out from the diagram?

Look at all the information you've now got about the triangles. What is the minimum you need to establish similarity?

What do you need to find out? What math do you know that connects to that?

What's the same about these two diagrams? What is different?

Support students in developing written explanations. Suggest that they write rough notes on their mini-whiteboards as they make decisions. Suggest that they refer to these notes as they write a fuller, clearer version of the explanation on their poster. Suggest that they are careful to name objects so that the reader understands what they are writing about.

If any students finish early, they should be encouraged to use the *Blank Cards* provided to create some new problems for others to work on.

Sharing work (10 minutes)

As students finish their posters, get them to critique each other's work by asking one student from each group to visit another group. Students who visit another group should first jot down a list of which diagrams are in which category (e.g. Similar: A, D, E etc.) on their mini-whiteboards.

Use slide P-4 of the projector resource, to explain how students should work together:

Sharing Work

1. **One** person in your group record your results on your mini-whiteboard and then go to another group's desk and check your work against theirs.
2. If there are differences, ask for an explanation.
   If you disagree, explain which column you think is correct.
3. If you have categorized diagrams in the same columns, compare your methods and check that you understand each others' explanations.
4. If you are staying at your desk, be ready to explain the reasons for your group's decisions.

Poster Review (10 minutes)

Once students have had a chance to share their work, and discuss their categorizations and reasoning with their peers, give them a few minutes to review their posters.

Now that you have discussed your work with someone else, as a group you need to consider whether or not to make any changes to your work.

If you think a diagram is in the wrong column, draw an arrow on your poster to show which column it should move to. If you are confident with your decisions, go over your work in pen (or make amends in pen if you have changed your mind.)

Whole-class discussion (15 minutes)

Organize a whole-class discussion about what has been learned and explore the different methods of justification used when categorizing diagrams.

You may want to first select a diagram that most groups categorized correctly, as this may encourage good explanations.

Which column did you put this diagram in? Can you explain your decision?

Can anyone improve this explanation?

Does anyone have a different explanation?

Which explanation do you prefer? Why?

Try to include a discussion of at least one diagram from each of the three columns.

Give me a diagram that shows triangles that are similar/not similar/where their similarity cannot be determined.

Why did you put this diagram in this column?

Did anyone put this diagram in a different column?

Once one group has justified their choice for a particular diagram, ask other students to contribute ideas of alternative approaches and their views on which reasoning was easier to follow. To help students explain their work, there are slides in the projector resource showing each of the diagrams A to H from the lesson task (Slides P-5 to P-12). When these are projected students may need to add their labels to the vertices to help their explanations.

Ask students what they learned by looking at other students' work and whether or not this helped them with diagrams they had found difficult to categorize or were unsure about:

Which diagram did you find the most difficult to categorize? Why do you think this was?

Did seeing where another group had placed this diagram help? If so, in what way did it help?

In what ways did having another group critique your poster help?

Did looking at another group's poster help you with your own work? Can you give an example?

During the discussion, draw out any issues you noticed as students worked on the activity, making specific reference to the misconceptions you noticed. You may want to use the questions in the *Common issues* table to support your discussion.

Follow-up lesson: reviewing the assessment task (15 minutes)

Give each student a copy of the review task, *Puzzling Triangles (revisited)* and their papers from the original assessment task, *Puzzling Triangles*. If you have not added questions to individual pieces of work then write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work. Some teachers give this as homework.

Look at your original responses and the questions [on the board/written on your script.]

Use what you have learned to answer the questions.

Now look at the new task sheet, Puzzling Triangles (revisited). Can you use what you have learned to answer these questions?

SOLUTIONS

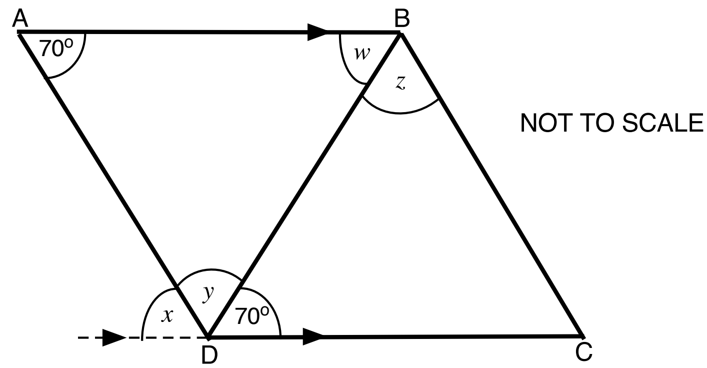
The solutions below are examples of correct answers to the questions. There are frequently other approaches. Encourage students who find this work easy to produce the most elegant solutions they can.

Assessment Task: Puzzling Triangles

1. The transversals AD and BC cannot be assumed to be parallel. Students may assume that they are, leading to incorrect reasoning. It is important that students identify a mathematical structure from which to derive their numerical answers.

Angle BDC and w are alternate interior angles formed by a transversal crossing a pair of parallel lines and so are equal. Therefore $w = 70°$.

Similarly, angle BAD and angle x are alternate interior angles formed by a transversal crossing a pair of parallel lines and so are equal, so $x = 70°$ also.



Angles x, y and angle BDC are on a straight line so they sum to 180°. So $y = 180° - x - 70° = 180° - 70° - 70° = 40°$. (The sum of the angles in a triangle may also be used to show that $y = 40°$, once angle w has been calculated.)

The measure of angle z cannot be deduced from the diagram, as the transversals cannot be assumed to be parallel.

2. All of the angles in the two triangles can be calculated to show that the two triangles are not similar. The missing angle in the first triangle is 45° and the three angles in the second triangle are 40°, 70° and 70°. It is, however, not necessary to make all these calculations before concluding that the triangles are not similar. For example, angle DAB is 110˚ (corresponding on parallel lines) and hence BAC is 40˚. Angle ABC is 70˚ (angles on a straight line) so angle ACB cannot be 65˚ therefore the triangles cannot be similar.

3. The two triangles have a shared angle (BCD) and they each have an angle of 65° (Angle CBE and angle CEA). If two of the angles are the same the third angle must also be the same so the two triangles are similar.

Sorting Triangles

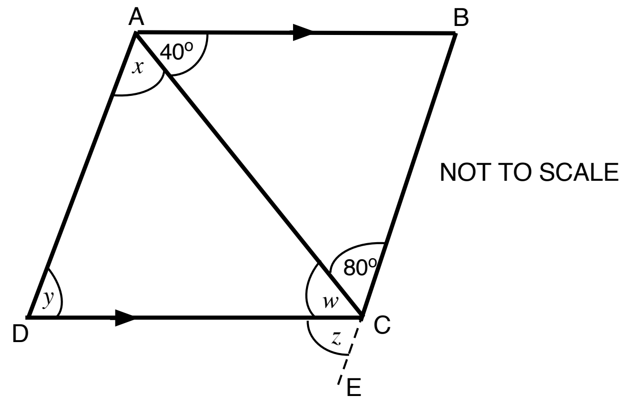
The quality of the written reasoning with references to angle theorems and triangle properties is the focus of this task. However, we provide a list of categorizations for your convenience.

Note that students may make alternative assumptions and this may lead to different results. For example, some may argue that one cannot assume that the apparently straight lines on the diagrams are in fact straight!

Similar	Not similar	Cannot be determined
A <SRP=55° (isosceles triangle) <SPR = 70° (angles in triangle = 180°) <QPR = 55° (alternate angles) So two angles in each triangle are equal. D < STR=70° (corresponding angles) <QPS = <TSR (corresponding angles) So two angles in each triangle are equal. E <PQR = <SRT (alternate angles) So two angles in each triangle are equal. H <QPR = <RST (alternate angles) <PRQ = < SRT (vertically opposite angles) So two angles in each triangle are equal.	B <QRS = 70° (isosceles triangle) <SQR = 40° (angles in triangle = 180°) <PQS = 70° (alternate angles) <QPS = <QSP = 55° (base angles of isosceles triangle) So as triangle QSR does not contain an angle of 55° the triangles are not similar. G <QRS = 70° (alternate) So triangle QRS is isosceles. (Base angles both 70°) < SQR = 40° (Angles in a triangle = 180°) If triangle PRS was similar to triangle QRS, then at least one of the two angles PRS or PSR would be 70° in which case P would be coincident with Q. As P and Q are distinct, the two triangles cannot be similar.	C We cannot assume lines RQ and US are parallel, so all that can be said is that <QPR = <STU (alternate). F We, cannot assume that PR is parallel to TS and, even if QRS is a straight line, all that can be said is that < PQR = < TRS.

Assessment Task: Puzzling Triangles (revisited)

1. The angle BAC and *w* are alternate interior angles formed by a transversal crossing a pair of parallel lines and so are equal. Therefore $w = 40°$.



Angles *z*, *w* and angle BCA are on a straight line so they sum to 180°.

So $z = 180° - w - 80°$
$= 180° - 40° - 80°$
$= 60°$.

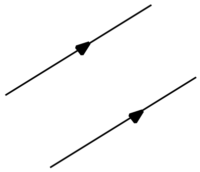
The measures of angle *x* and *y* cannot be deduced from the diagram.

2. Using alternate angles gives angle ABC as 70° in total. As angle CBD is 35˚, this means that angle ABD must also be 35˚ and so both triangles ABD and CBD contain a 35° angle. Triangle ABC is isosceles so angle BAD is equal to angle BCD (it can be calculated but this is not necessary). If two of the angles are the same the third angle must also be the same so the two triangles are similar. (In fact they are congruent).

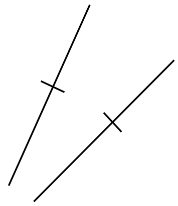
3. All of the angles in the two triangles can be calculated to show that the two triangles are not similar. The missing angles in triangle CDE are 40° and 70°. The three angles in triangle ABC are 30°, 70° and 80°. Alternatively, angle DEC is 70˚ (vertically opposite), so angle DCE is 40˚ (angles in triangle CDE sum to 180˚). However, angle ABC is 80˚ (angles on a straight line). The two triangles cannot be similar because one has an angle of 80˚ and the other does not.

Puzzling Triangles

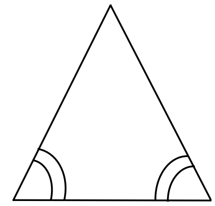
Use the diagrams below to remind you of markings and their meanings:



These lines are parallel



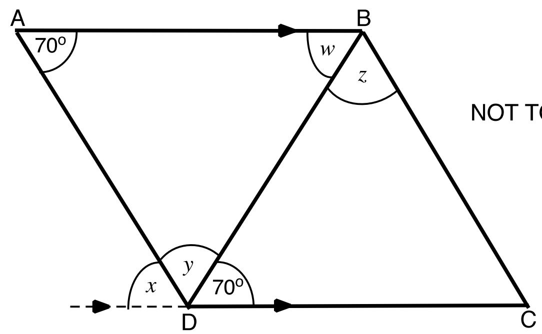
These lengths are congruent



These angles are congruent

1. In this diagram two 70° angles have been labeled. Four other angles are labeled w, x, y and z.

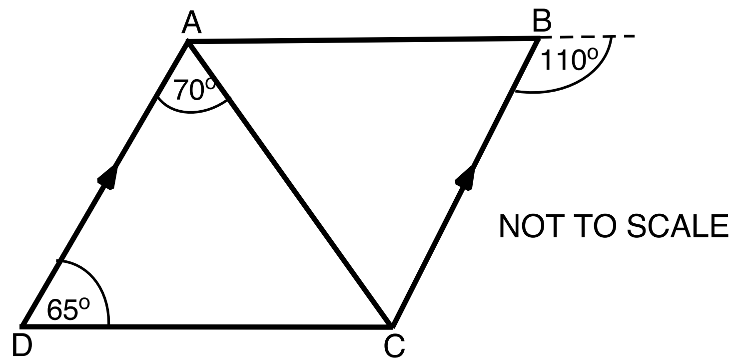
Complete the table below. If it is possible to work out the measure of the angle, write it down and show your method. If it is not possible, explain how you know.



NOT TO SCALE

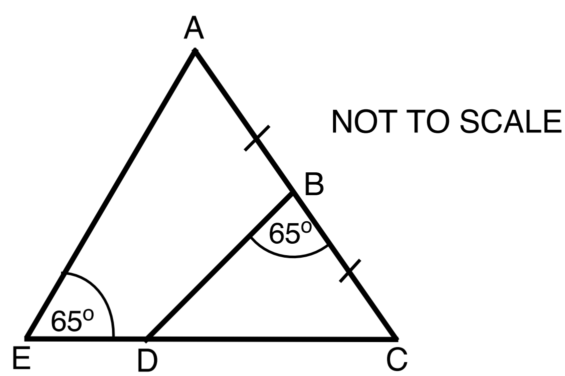
Angle	Is it possible to find the measure of the angle? If so, what is it?	Explain carefully how you decided.
w		
x		
y		
z		

2. The diagram below shows two triangles, triangle ABC and triangle ACD:



Are the two triangles similar? Explain, step by step, how you know.

3. The diagram below shows triangle BCD within triangle ACE.

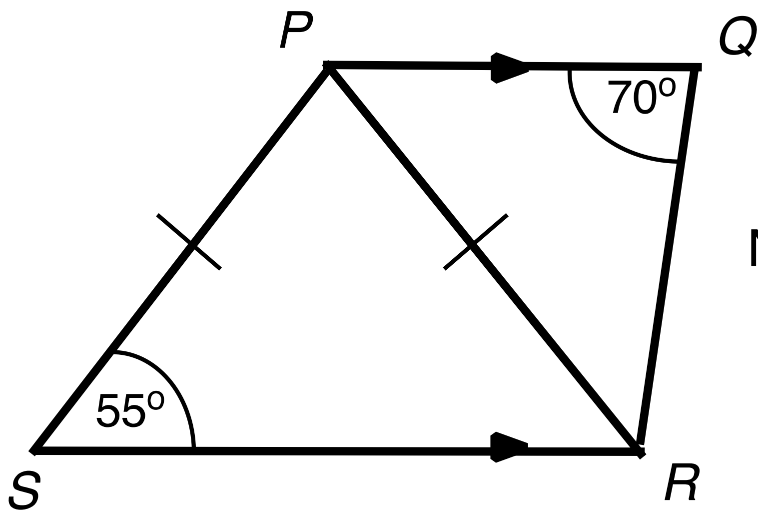


Are the two triangles similar? Explain, step by step, how you know.

Sorting Triangles

A

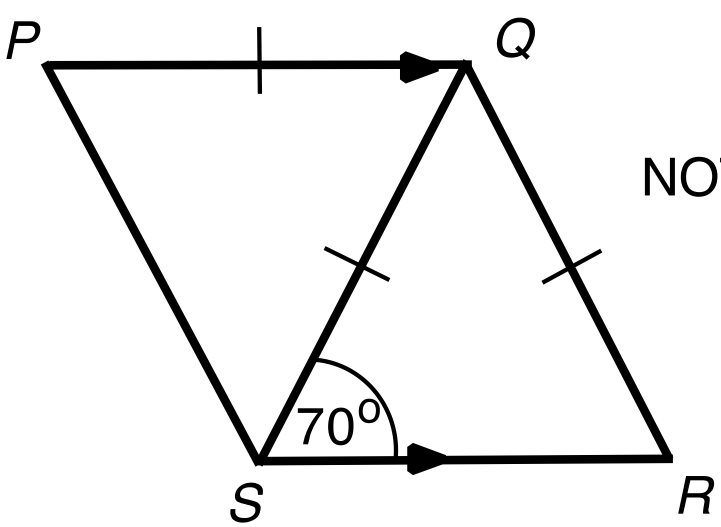
Are triangles *PQR* and *SPR* similar?



NOT TO SCALE

B

Are triangles *PQS* and *QRS* similar?

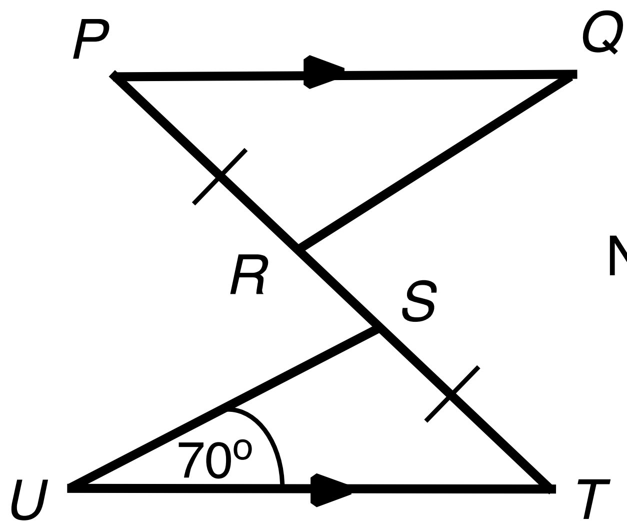


NOT TO SCALE

Sorting Triangles (continued)

C

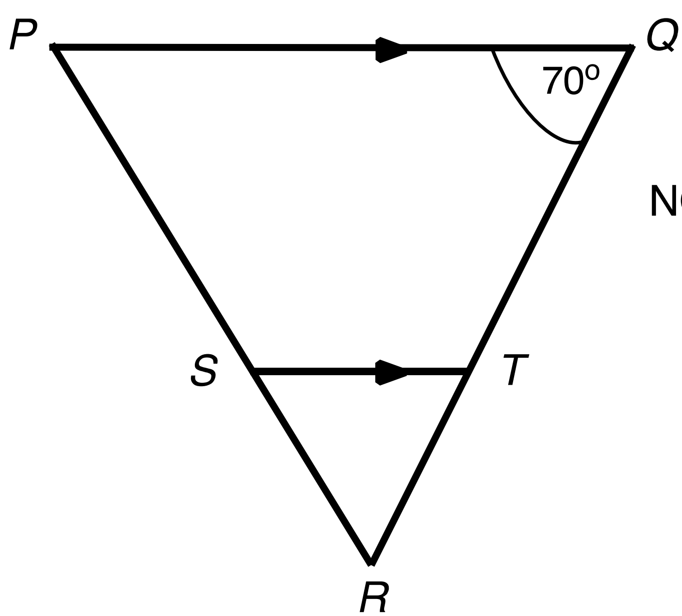
Are triangles *PQR* and *STU* similar?



NOT TO SCALE

D

Are triangles *PQR* and *STR* similar?

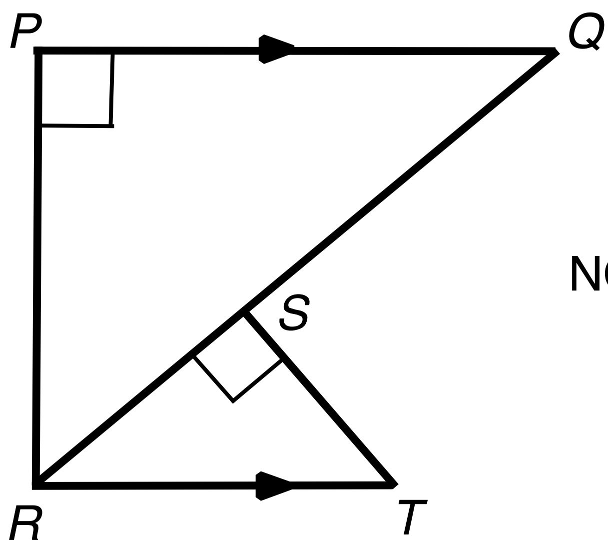


NOT TO SCALE

Sorting Triangles (continued)

E

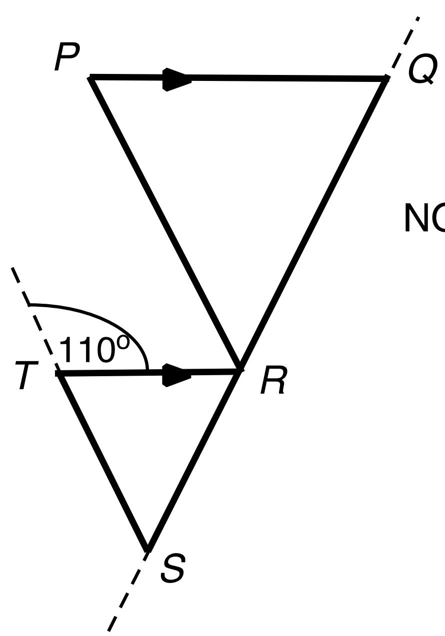
Are triangles *PQR* and *STR* similar?



NOT TO SCALE

D

Are triangles *PQR* and *TRS* similar?

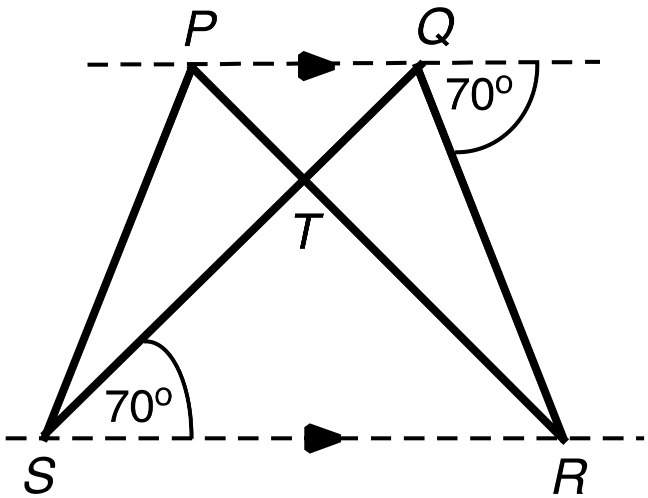


NOT TO SCALE

Sorting Triangles (continued)

G

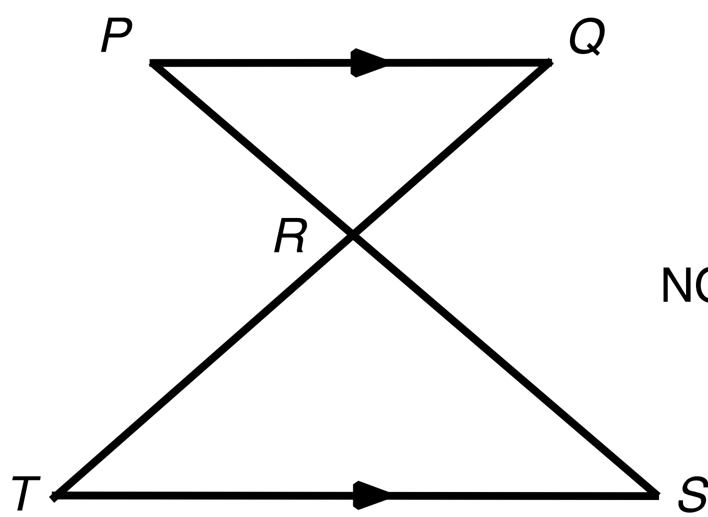
Are triangles *PRS* and *QRS* similar?



NOT TO SCALE

H

Are triangles *PQR* and *RST* similar?



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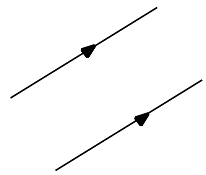
Blank Cards

My example

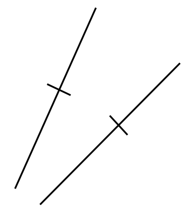
My example

Puzzling Triangles (revisited)

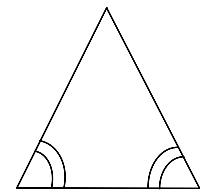
Use the diagrams below to remind you of markings and their meanings:



These lines are parallel



These lengths are congruent



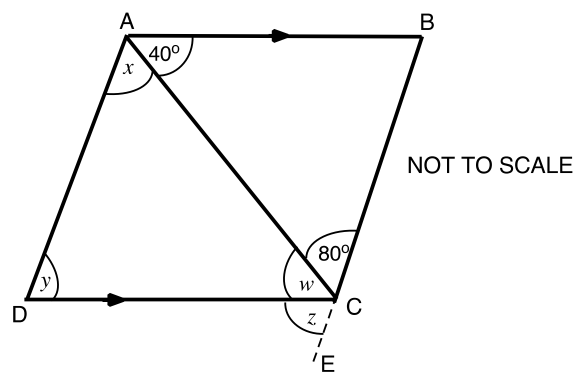
These angles are congruent

1. In this diagram an 80° angle and a 40° angle have been labeled.

   Four other angles are labeled *w, x, y* and *z*.

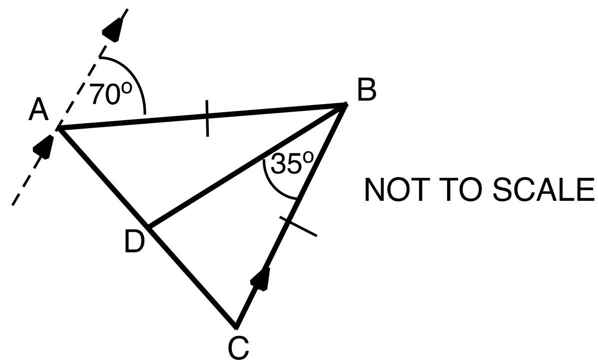
   BCE is a straight line.

   Complete the table below. If it is possible to work out the measure of the angle, write it down and show your method. If it is not possible, explain how you know.



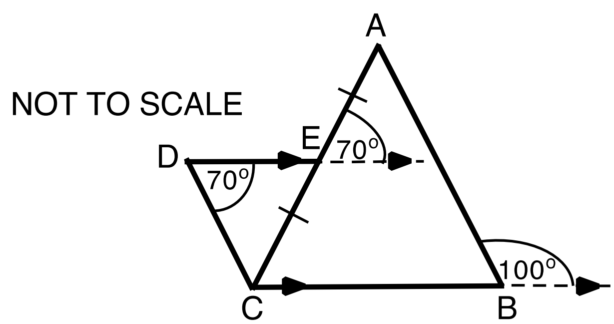
Angle	Is it possible to find the measure of the angle? If so, what is it?	Explain carefully how you decided.
w		
x		
y		
z		

2. The diagram below shows triangle ABD and triangle CBD:



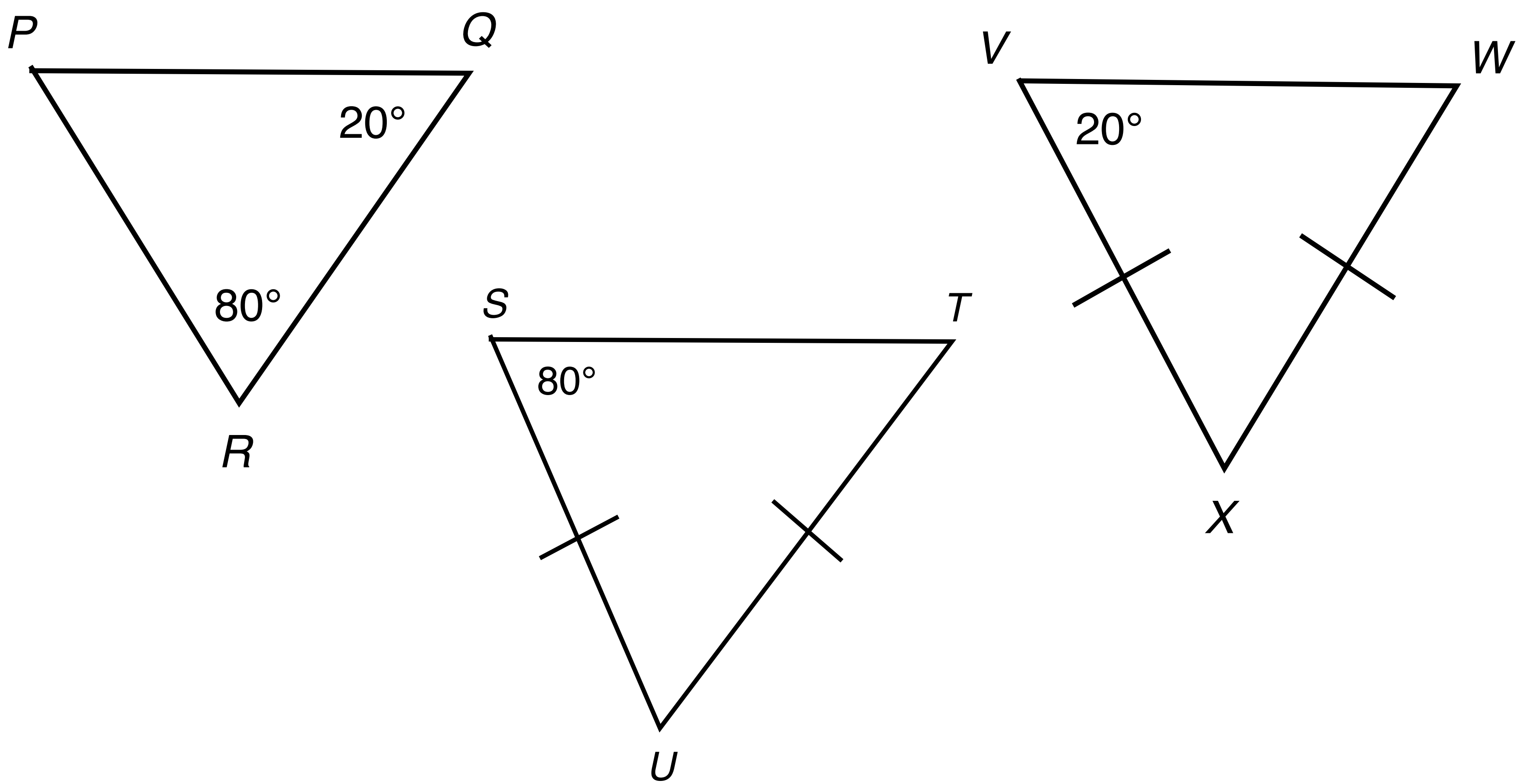
Are the two triangles similar? Explain, step by step, how you know.

3. The diagram below shows triangle ABC and triangle CDE. E is the midpoint of AC.



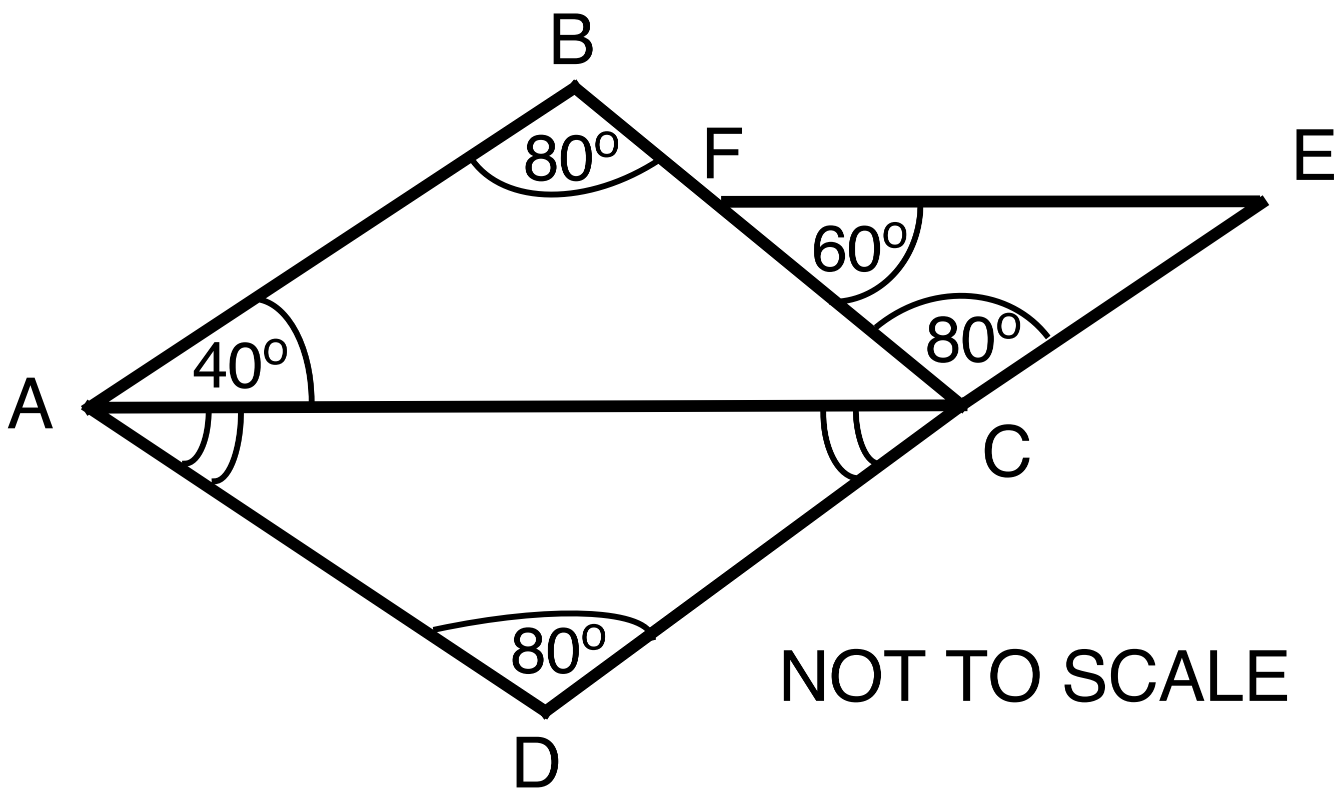
Are the two triangles similar? Explain, step by step, how you know.

Are these Mathematically Similar?



Checking for Similarity

Are any of the triangles ABC, CEF and ACD mathematically similar?



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Working Together

Take turns to:

1. Select a diagram, and decide whether or not the two triangles are similar, not similar, or ‘it cannot be determined’.
   Explain your reasoning.

2. Fill in as many angles as you can.

3. Your partner must challenge your explanation if they disagree or describe it in their own words if they agree.

4. Once agreed, glue the diagram onto the poster and write your explanation **in pencil** next to it.

5. Continue to take turns until all the diagrams are sorted.

Sharing Work

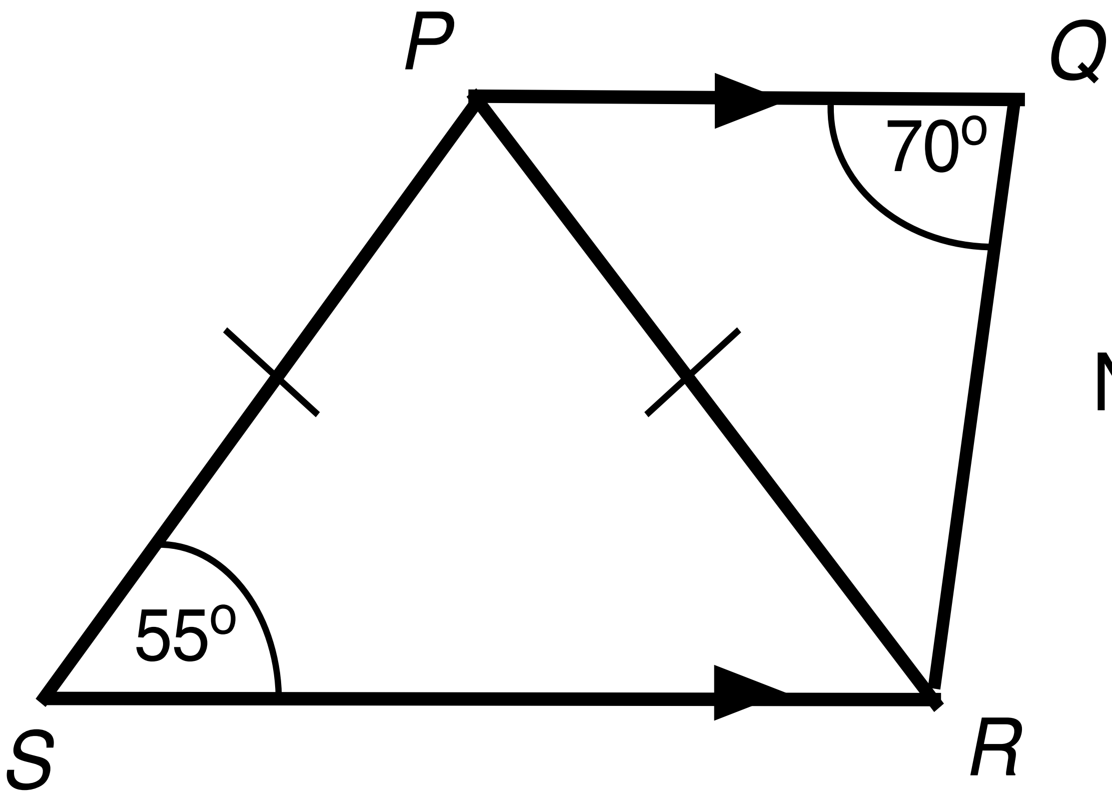
1. **One** person in your group record your results on your mini-whiteboard and then go to another group's desk and check your work against theirs.

2. If there are differences, ask for an explanation.
   If you disagree, explain which column you think is correct.

3. If you have categorized diagrams in the same columns, compare your methods and check that you understand each others' explanations.

4. If you are staying at your desk, be ready to explain the reasons for your group's decisions.

Are triangles *PQR* and *SPR* similar?



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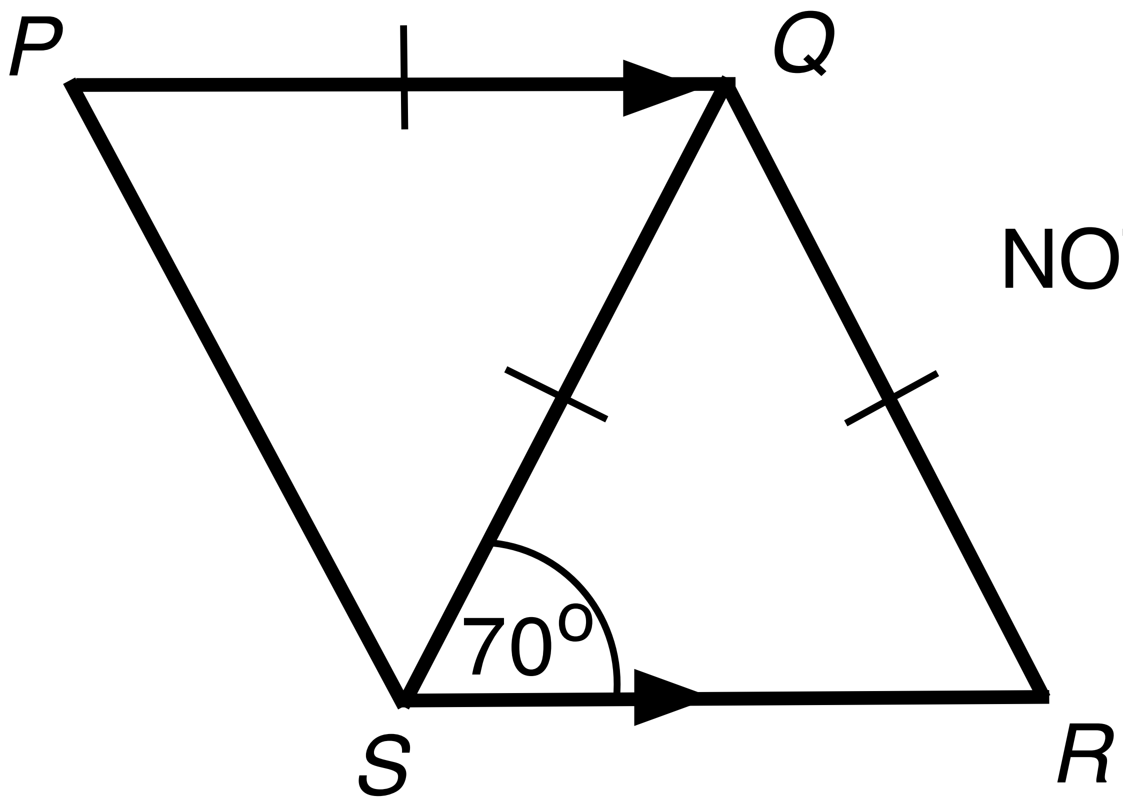
Are triangles *PQS* and *QRS* similar?



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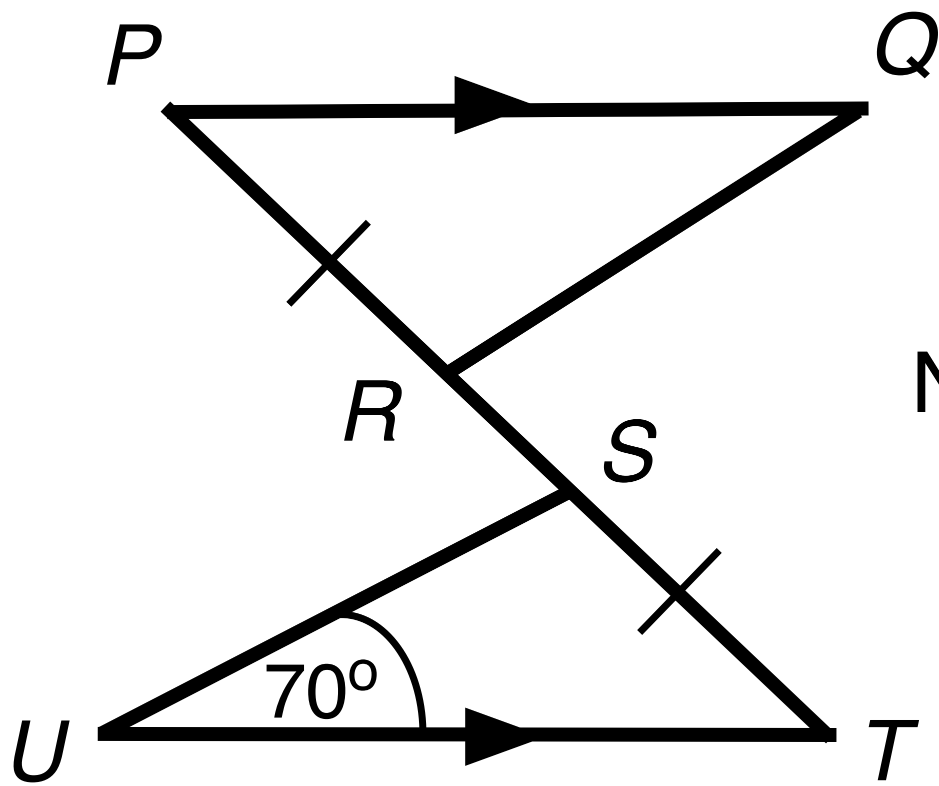
Are triangles *PQR* and *STU* similar?



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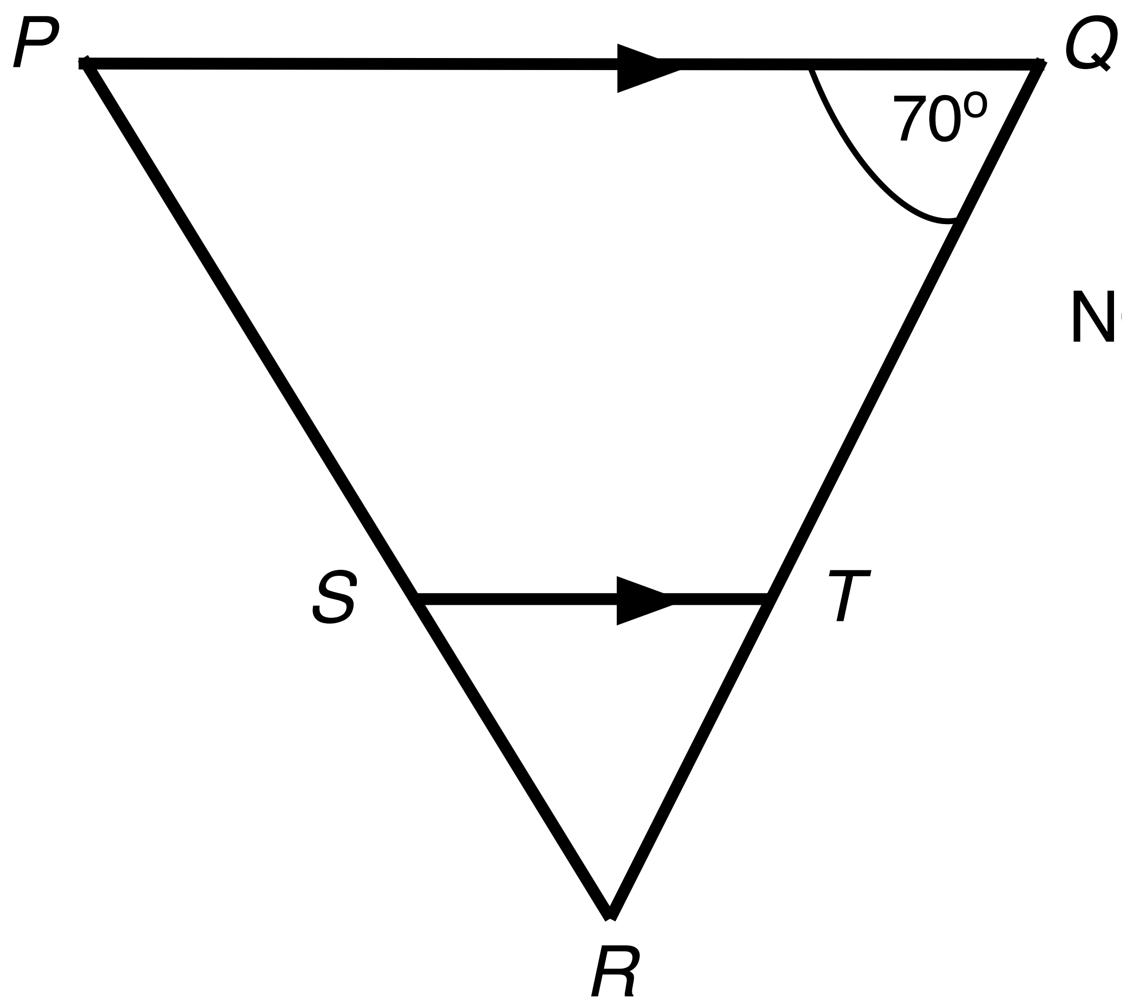
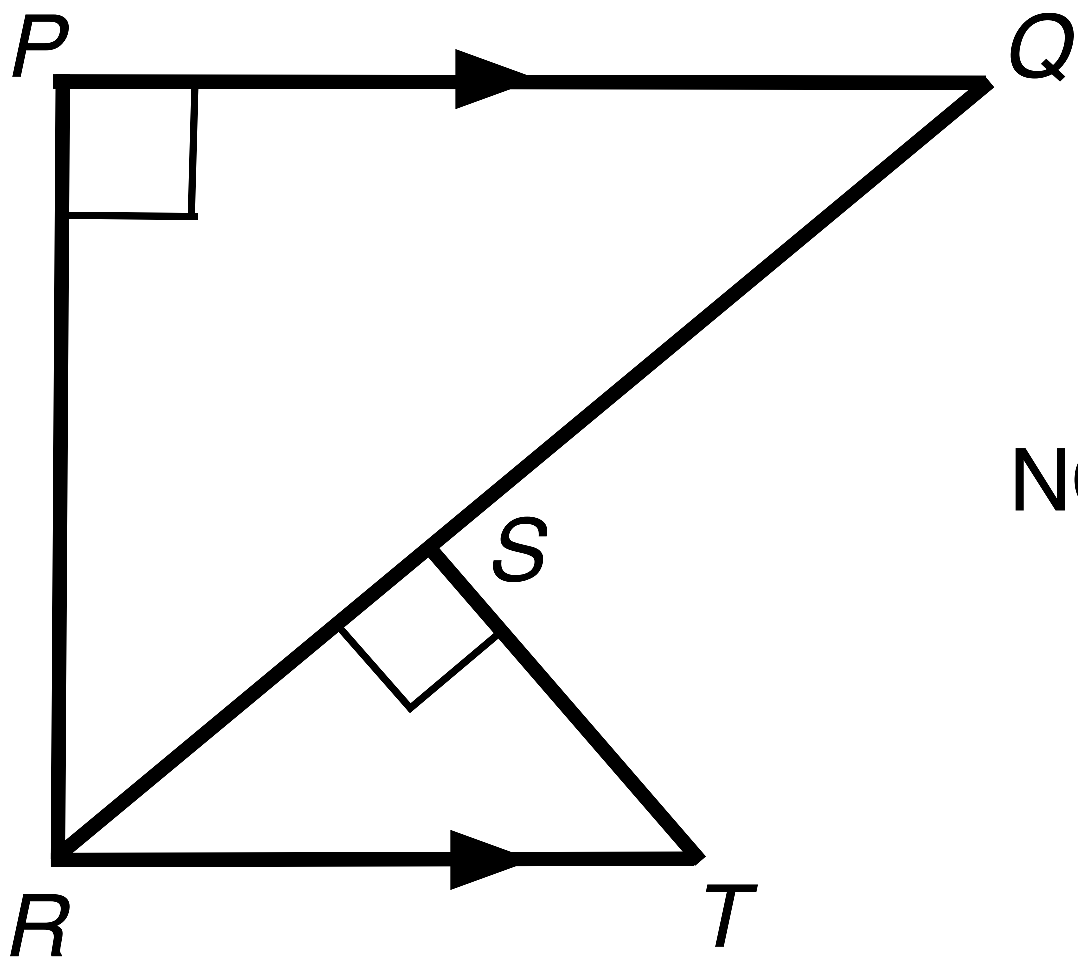
Are triangles *PQR* and *STR* similar?



NOT TO SCALE

Are triangles *PQR* and *STR* similar?



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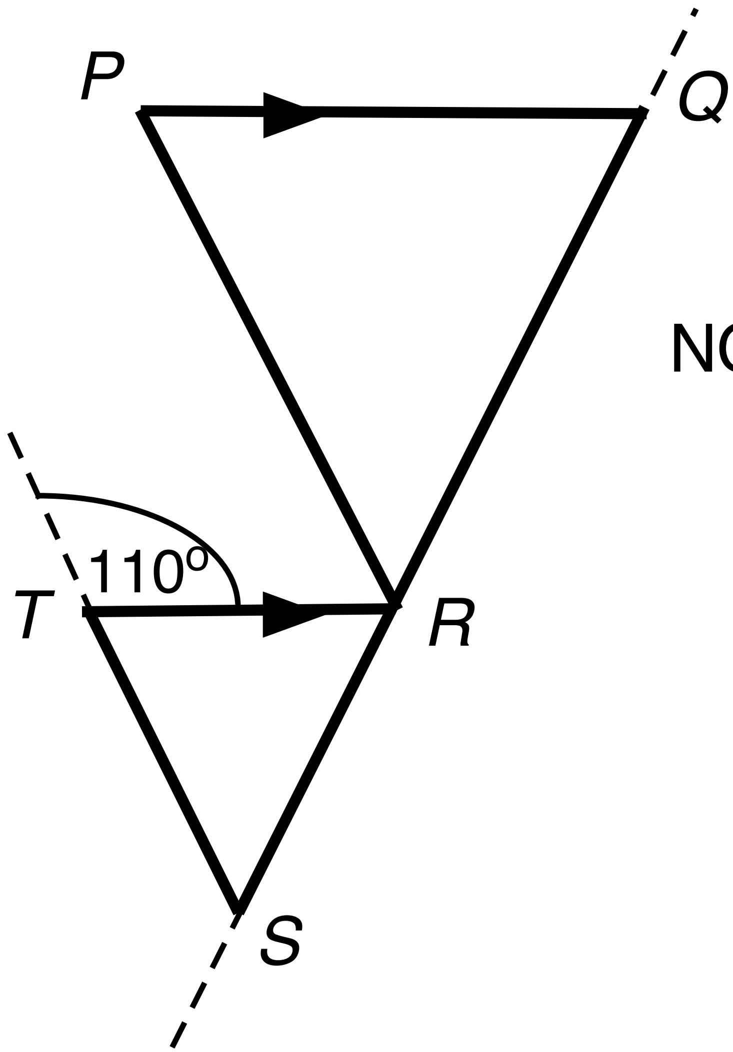
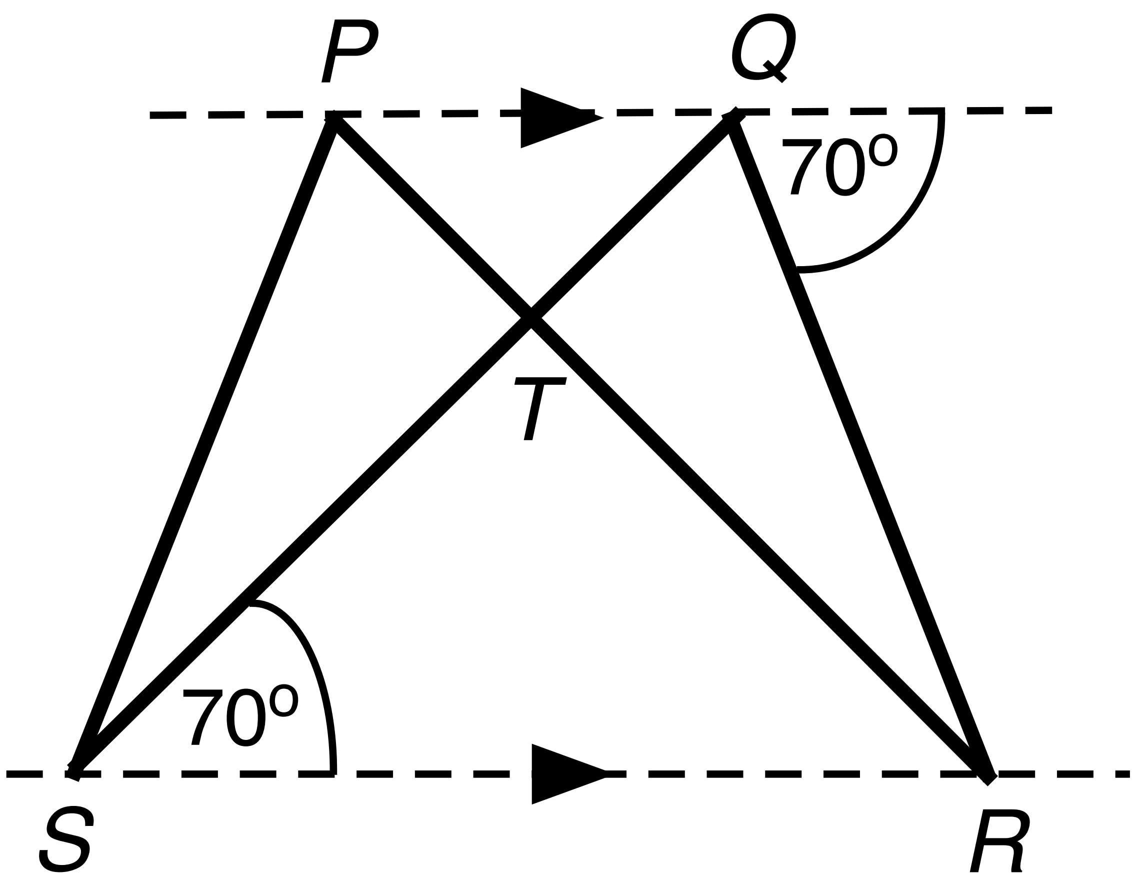
Are triangles *PQR* and *TRS* similar?



NOT TO SCALE

Are triangles *PRS* and *QRS* similar?



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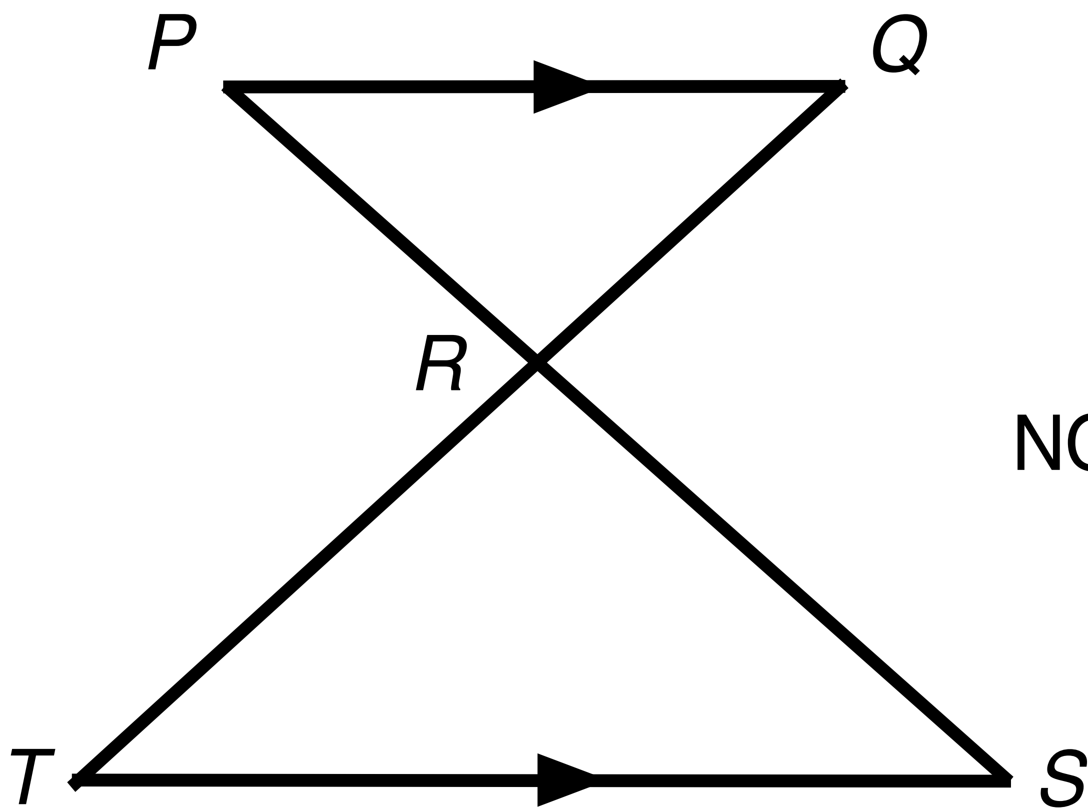
Are triangles *PQR* and *RST* similar?



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Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
Shell Center Team at the Center for Research in Mathematical Education
University of Nottingham, England:

Malcolm Swan,
Nichola Clarke, Clare Dawson, Sheila Evans, Colin Foster, and **Marie Joubert**
with
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Hugh Burkhardt, Daniel Pead, and **Malcolm Swan** at the University of Nottingham

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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org