CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Discovering the Pythagorean Theorem

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Discovering the Pythagorean Theorem

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Use the area of right triangles to deduce the areas of other shapes.
- Use dissection methods for finding areas.
- Organize an investigation systematically and collect data.
- Deduce a generalizable method for finding lengths and areas (The Pythagorean Theorem.)

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards* for Mathematics:

8.G: Understand and apply the Pythagorean Theorem.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 1, 2, 3, 7 and 8:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

- Before the lesson, students attempt a task individually. You review their work and create questions for students to answer in order to improve their solutions.
- A whole-class introduction poses the problem of finding the areas of 'tilted squares' drawn on a square grid. Students share different approaches for calculating areas and are offered three generalizable methods that they might use. Students are asked to find possible areas of 'tilted squares' at a specific tilt. This requires a systematic approach.
- In a whole-class discussion results are shared and organized.
- In a follow-up lesson, students receive your comments on the assessment task and use these to attempt the similar task, approaching it with insights gained from the lesson.

MATERIALS REQUIRED

- Each student will need a copy of the task sheets *Square Areas, Tilted Squares, Proving the Pythagorean Theorem, Square Areas* (*revisited*), and multiple copies of the Dotted Grid Paper (on demand).
- Each small group of students will need copies of the sheet *Some Different Approaches*.
- There are projector resources to help introduce activities and support whole-class discussions.

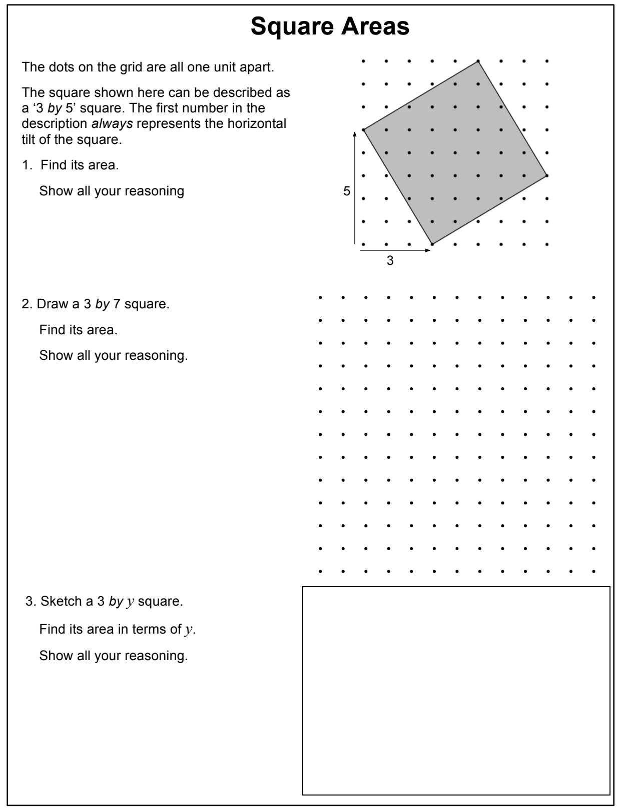
TIME NEEDED

20 minutes before the lesson, an 80-minute lesson (or split into two shorter lessons), and 20 minutes in a follow-up lesson (or for homework.) Timings given are only approximate. Exact timings will depend on the needs of your class.

BEFORE THE LESSON

Assessment task: *Square Areas* (20 minutes)

Ask students to do this task, in class or for homework, a day or more before the lesson. This will give you an opportunity to assess their work, and to find out the kinds of difficulties they have. You will then be able to target your help more effectively in the next lesson.


Square Areas

The dots on the grid are all one unit apart.

The square shown here can be described as a '3 *by* 5' square. The first number in the description *always* represents the horizontal tilt of the square.

1. Find its area.

   Show all your reasoning

2. Draw a 3 *by* 7 square.

   Find its area.

   Show all your reasoning.

3. Sketch a 3 *by* y square.

   Find its area in terms of y.

   Show all your reasoning.

If you think your students will have a problem using dot paper, spend time discussing how to measure lengths and areas on the paper. Use Slide P-1 of the projector resources to project a grid.

Each student will need a ruler, a pencil, and a sheet of squared dot paper for their rough work.

Now explain what you are asking students to do.

> *Spend 15 minutes on your own answering these questions as carefully as you can. Show all your work so that I can understand your reasoning. There will be a lesson on this material [tomorrow] that will help you improve your work.*

Project the dotted squares on the board and check that students understand the notation *x by y* for describing the tilted squares. Ask them to respond to questions such as the following using their mini-whiteboards:

> *Can you draw me a 2 by 1 square?*
>
> *Can you draw me a 3 by 2 square?*
>
> *The two numbers tell you where one side of the square is. The first number tells you the horizontal distance from a point on the grid to a vertex of the square, the second number tells you the vertical distance from the same point on the grid to a second vertex of the square.*

It is important that, as far as possible, students are allowed to answer the task questions without assistance. If students are struggling to get started then ask questions that help them understand what is required, but make sure you do not do the task for them.

When all students have made a reasonable attempt at the task, tell them that they will have time to revisit and revise their solutions later.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their problem solving strategies.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a list of questions. Some suggestions for these are given in the *Common issues* table on the next page. We suggest that you make a list of your own questions, based on your students' work, using the ideas in the table. We recommend you write one or two questions on each student's work. If you do not have time to do this, you could:

- Select a few questions that will be of help to the majority of students, and write these questions on the board when you return the work to the students.
- Or give each student a printed version of your list of questions and highlight the questions for each individual student.

Common issues:	**Suggested questions and prompts:**
Estimates the area of the square (Q1 and Q2) For example: The student divides the first diagram into squares and attempts to count them. Or: The student uses a ruler to measure the length of a side of the square. Or: The student determines there is a 4 by 4 square in the center of the shaded square. Then adds to this area an estimate for the areas of the remaining four triangles (Q1.)	• Do you think your method will give an exact answer? Why? • Can you think of a method that will give you a more precise answer? • Can you find a way of calculating the area without counting squares? • Dots are one unit apart. Can you find the area of the square in square units?
Dissects the square into smaller shapes (e.g. triangles), but these do not permit an accurate calculation (Q1 and Q2)	• Can you find a method of dividing the square into smaller triangles, for which you know the base and the height of each triangle? • Can you find a method where you don't have to divide the square up into smaller pieces?
Uses Pythagorean theorem to figure out the areas	• Can you find a precise way of calculating the area without using the Pythagorean theorem?
Has difficulty with the concept of variable. (Q3) For example: The student assigns a particular value to y, for example 9 as this forms a pattern with previous questions.	• y can be any value. Make sure your answer for the area includes y. • In questions 1 and 2, what were the values of y? • What would the area be if y was 10? 100? Show your method using y to represent other numbers like these.
Has difficulty with algebraic notation (Q3)	• Can you use your method from question 1 to calculate the area?
Completes the task successfully	• What other areas can you make by drawing tilted squares on the dotted grid? • What areas are impossible to make? Can you show that these are impossible?

SUGGESTED LESSON OUTLINE

Interactive Introduction: *Tilted Squares* (10 minutes)

Give students the sheet *Tilted Squares*.

Today we are going to do some more questions like the one I gave you yesterday. This time, you are going to work in pairs, sharing your ideas.

Allow students a few minutes to solve the first question individually, then discuss the question with a partner.

As they do this, go around the class and notice the different methods for finding the areas of the shapes. Use your findings from the assessment to help you find students who may have different approaches. This information will help you structure the follow-up whole-class discussion.

You may want to use the questions in the *Common issues* table to help address difficulties.

If you think your students are struggling to understand the notation x by y for describing a tilted square, then ask:

Draw me a 2 by 3 square.

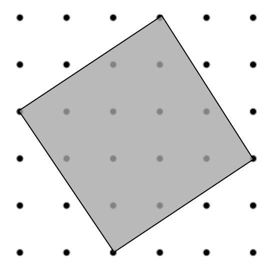
Draw me a 3 by 2 square.

Whole-class discussion to share students' own approaches (10 minutes)

Use the data projector to project the grid on Slide P-1 and draw a large version of the tilted 2 by 3 square on the board.

This tilted square has a x-value of 2 and a y-value of 3.

Have any of you got a good method for finding the area? Come out and show me on the board.



Ask students to share their methods with the whole class. Choose pairs of students that you know have different approaches. If possible, include one that involves counting squares and one that involves dissection. Focus on getting students to understand these approaches fully before criticizing or refining them.

Can you explain in your own words how (Marla) approached the task?

Then, each time, ask:

Do you think that this method will give an accurate answer? Why?

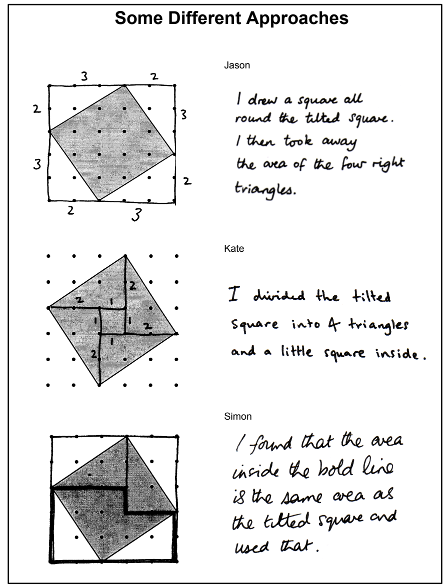
Do you think this method is quick and easy to use? Why?

Whole class discussion on the sample student work (15 minutes)

Give out the sheet: *Some different approaches*.

Allow students a few minutes to read through the methods used by Jason, Kate, and Simon then organize a whole-class discussion of these approaches.

Use Slides P-2, P-3, and P-4 of the projector resource to show their diagrams on the board.



These might be helpful approaches.
Can you use these methods to find the areas?

Do you think they will give an accurate answer?
Why or why not?

Do you agree with Simon's statement that the area inside the bold line is the same as the area of the tilted square?

Jason's method leads to:

Area of square – area of four triangles $= (2+3)^2 - 4\times\frac{1}{2}\times 2\times 3 = 13$

Kate's dissection leads to:
Area of four triangles + square in the middle

$= 4\times\frac{1}{2}\times 3\times 2 + (3-2)^2 = 13$

Simon's method leads directly to: Area of tilted square = 3^2+2^2 = 13

Extending the lesson over two days

If you are taking two days to complete the unit then you may want to end the first lesson here. At the start of the second day, briefly remind students of their previous work before moving on to the next collaborative activity.

Working in pairs: Finding possible areas (15 minutes)

Have extra copies of the *Dotted Grid Paper* to give students.

Introduce the task:

In your pairs, you are now going to see if there is a quick way to figure out the areas of tilted squares.

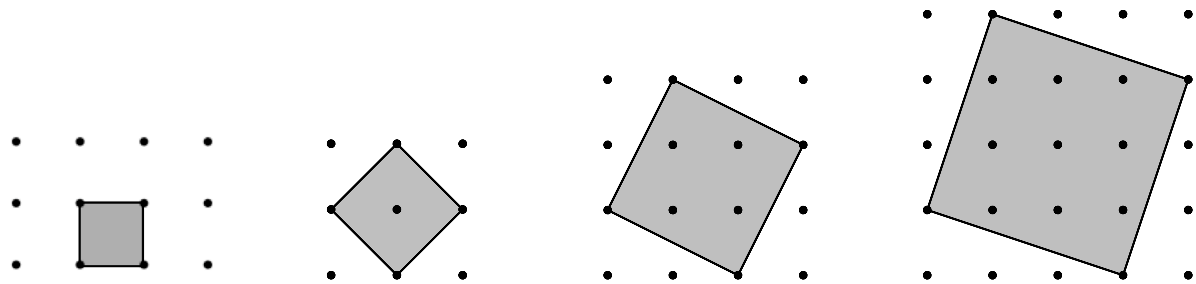
Each group is to investigate the area of squares with a fixed y-value.

See if you can come up with a method that you can use for all your squares.

In trials it was found that many students were unable to organize a systematic approach for themselves. We suggest, therefore, that you ask each pair of students to work on squares with a specific y-value.

For example, one pair may work with squares with a y-value 0, another pair, squares with a y-value of 1, and so on, up to squares of a y-value of 4. It is likely that you will have several pairs working on squares with the same y-value. If a group finishes quickly they should start on a different y-value.

For example, the diagrams below show squares of a y-value of 1, y is fixed at 1 unit and x is increased by 1 each time, starting with $x = 0$.



While students work in pairs you have two tasks: to note different student approaches to the task and to support student reasoning. Listen and watch students carefully as this will help you understand their thinking and help focus a whole-class discussion towards the end of the lesson.

Use the questions in the *Common issues* table to help address difficulties.

Prompt students with questions like:

Try keeping one corner of the square fixed, and move a second corner one unit along each time.

How can you find the areas?

Can you use the same method for calculating areas each time?

Can you see a pattern in your results?

For example, when the y-value is 1, as in the diagrams on the previous page, students may notice that the sequence of areas forms a pattern: 1, 2, 5, 10, 17, 26, …

What do you notice about this sequence of numbers? [Each is one greater than a square number.]

If students just notice the sequence of areas increase by 1, 3, 5, 7, 9 etc. encourage them to investigate how each square area relates to the x and y-values of the square.

Whole-class discussion: Organizing and generalizing class results (15 minutes)

Show Slide P-5 of the projector resource:

x / y	0	1	2	3	4	Comments
0						
1						
2						
3						
4						

Collect results from the different pairs of students and assemble them into the table. If there are still gaps, allocate different members of the class with squares to draw until you have about 5 rows and 4 columns.

Ask students if they have figured out a number pattern for each y-value. Record these on the board.

You may end up with a table like this:

x / y	0	1	2	3	4	Comments
0	0	1	4	9	16	They are all square numbers, x^2
1	1	2	5	10	17	They are all one more than square numbers, $x^2 + 1$
2	4	5	8	13	20	They are all four more than x^2 numbers, $x^2 + 4$
3	9	10	13	18	25	They are all nine more than x^2 numbers, $x^2 + 9$
4	16	17	20	25	32	They are all sixteen more than x^2 numbers, $x^2 + 16$

Encourage students to generalize and predict new rows of the table:

Can you tell me what the next row (a y-value of 5) will look like? [25, 26, 29, 34, … x^2+5^2]
How can you tell?

Some students will notice that the differences between terms vertically are successive odd numbers. Others may continue the pattern in the comments.

Ask the following questions in turn:

Can you tell me what the row with a y-value of 10 will look like? [100, 101, 104, 109, … x^2+10^2]

Can you tell me what is the area of a x by y square? [x^2+y^2]

Allow students a few minutes to think about each problem individually and then discuss the question with a partner.

Students may provide answers for a specific value of x, for example when $x = 0$ the area is y^2, or when $x = 1$ the area is $1+ y^2$, or when $x = 2$ the area is $4 + y^2$ and so on.

Encourage students to generalize further to a x *by* y square.

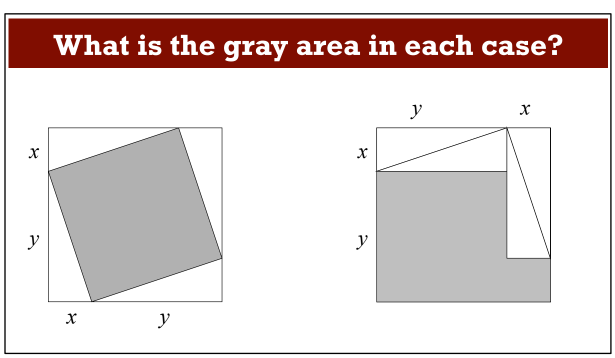
This exploration suggests that the area of the square will always be $x^2 + y^2$.

Whole class discussion: Proving the Pythagorean theorem (15 minutes)

Introduce the task:

We think that the area of the gray square is $x^2 + y^2$. But can we prove it?

Show Slide P-6 of the projector resource:



Ask the following questions in turn:

Can you explain to me why the two gray areas are equal?

What is the area of the second gray area? [$x^2 + y^2$] How can you see this?

So what is the area of the first gray area? [$x^2 + y^2$] Why?

This works for any right triangle with sides x and y.

What is the area of a (50, 100) tilted square?

Ask students to complete the sheet *Proving the Pythagorean Theorem* individually, to summarize the discussion.

Follow-up lesson: Reviewing the assessment task (20 minutes)

In the next lesson, give students their response to the original assessment task *Square Areas* and a copy of the task *Square Areas (revisited.)*

If you have not added questions to individual pieces of work, or highlighted questions on a printed list of questions, then write your list of questions on the board. Students should select only the questions from this list they think are appropriate to their own work.

Look at your original responses and the questions (on the board/written on your script.)

Use what you have learned to answer these questions.

Now have a go at the second sheet: Square Areas (revisited). Can you use what you have learned to answer these questions?

If you have not added questions to individual pieces of work or highlighted questions on a printed list of questions then write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

If you are short of time you could set this task as homework.

Suggestion for an extension activity

Is it possible to construct a square on the dotted paper with an area of 43? Can you prove this?

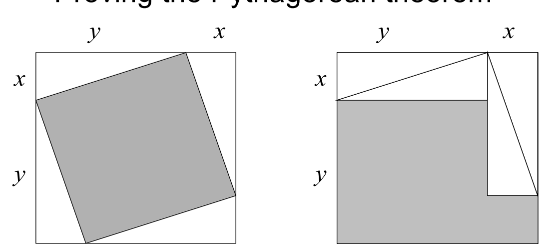
SOLUTIONS

Square Areas

1. The shaded area is given by $3^2 + 5^2 = 9 + 25 = 34$ square units.
2. The area of a (3, 7) square is $3^2 + 7^2 = 9 + 49 = 58$ square units.
3. The area of a (3, y) square is $3^2 + y^2 = 9 + y^2$ square units.

   The method used to deduce these answers may be similar to Jason's, Kate's or Simon's methods used in the lesson.

Proving the Pythagorean theorem



1. The area of the whole square is equal in both diagrams. The shaded area is this whole square minus four triangles in both diagrams. Thus, we are taking away equal quantities from equal quantities. So the shaded areas are equal.

2. The shaded area in the second diagram is easily seen as $x^2 + y^2$.

3. The hypotenuse of the right triangle with legs x and y is therefore $\sqrt{x^2 + y^2}$.

Square Areas (revisited)

1. The shaded area is given by $7^2 + 6^2 = 49 + 36 = 85$ square units.
2. The area of a (7, 5) square is $7^2 + 5^2 = 49 + 25 = 74$ square units.
3. The area of a $(7, y)$ square is $7^2 + y^2 = 49 + y^2$ square units.

   The method used to deduce these answers may be similar to Jason's, Kate's or Simon's methods used in the lesson or students may use the Pythagorean theorem.

Further notes on possible and impossible areas

It is possible to conjecture some patterns in these sequences. All square numbers leave a remainder of 0 or 1 when divided by 4. So sums of two square numbers (our areas) must leave a remainder of 0, 1, or 2 when divided by 4. This means that if a number takes the form $4n+3$ then it cannot be a tilted area. So this tells us that 3, 7, 11, 15, … cannot be tilted areas. But the list includes more areas than that.

Characterizing these sequences fully is not easy and it took one of the greatest mathematicians to do it. We are not suggesting that students are capable of this! However, for the interested:

Start by listing just the prime numbers in each list:

These *can* be made: 2, 5, 13, 17, 29, …

These *cannot* be made: 3, 7, 11, 19, 23, …

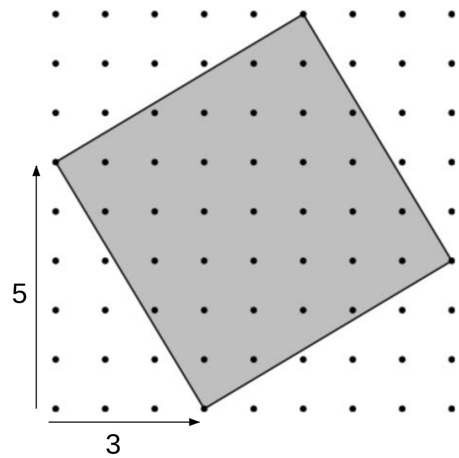
This indicates that, apart from 2, primes that are 1 more than multiples of 4 can be made while primes that are 3 more than multiples of 4 cannot be made.

- Fermat proved that a prime number p can be expressed as the sum of two squares if, and only if, p can be written as $4n+1$ where n is an integer.
- If two integers, x and y, can each be written as the sum of two squares, then their product, xy can be written as the sum of two squares. This just (!) involves some algebraic manipulation (easiest with complex numbers) to show that:
  $(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$.
- Combining these results, the theorem that completely solves the problem is: A number n is expressible as a sum of two squares if, and only if, in the prime factorization of n, every prime of the form $(4k+3)$ occurs an even number of times!

Square Areas

The dots on the grid are all one unit apart.

The square shown here can be described as a ‘3 *by* 5’ square. The first number in the description *always* represents the horizontal tilt of the square.



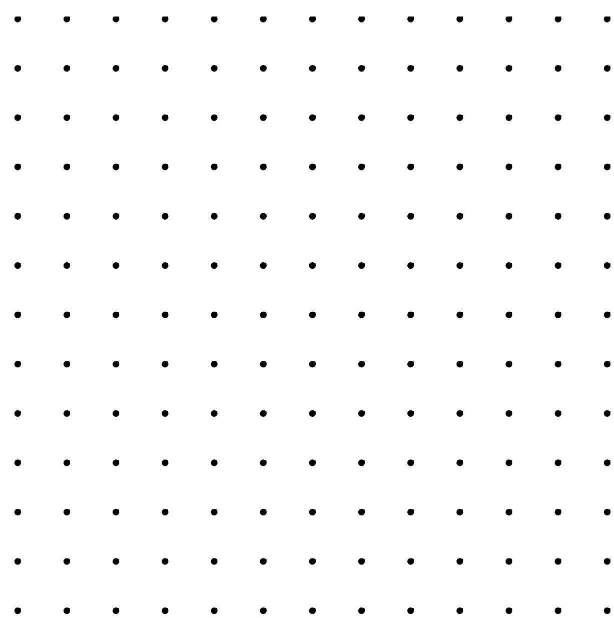
1. Find its area.

   Show all your reasoning

2. Draw a 3 *by* 7 square.

   Find its area.

   Show all your reasoning.



3. Sketch a 3 *by* y square.

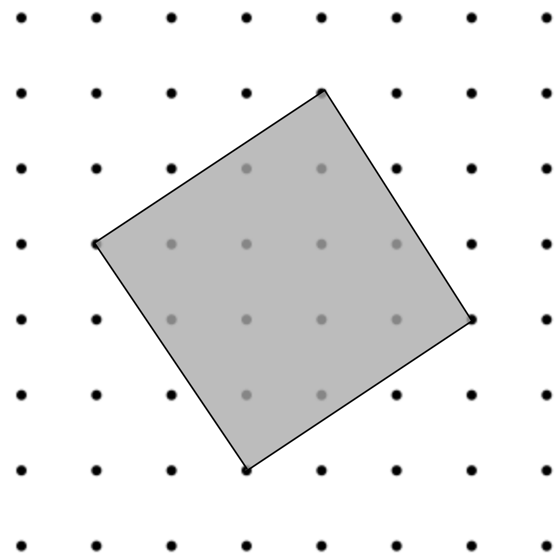
   Find its area in terms of y.

   Show all your reasoning.

Tilted Squares

1. Find the area of the 2 *by* 3 square.
   Show all your reasoning.

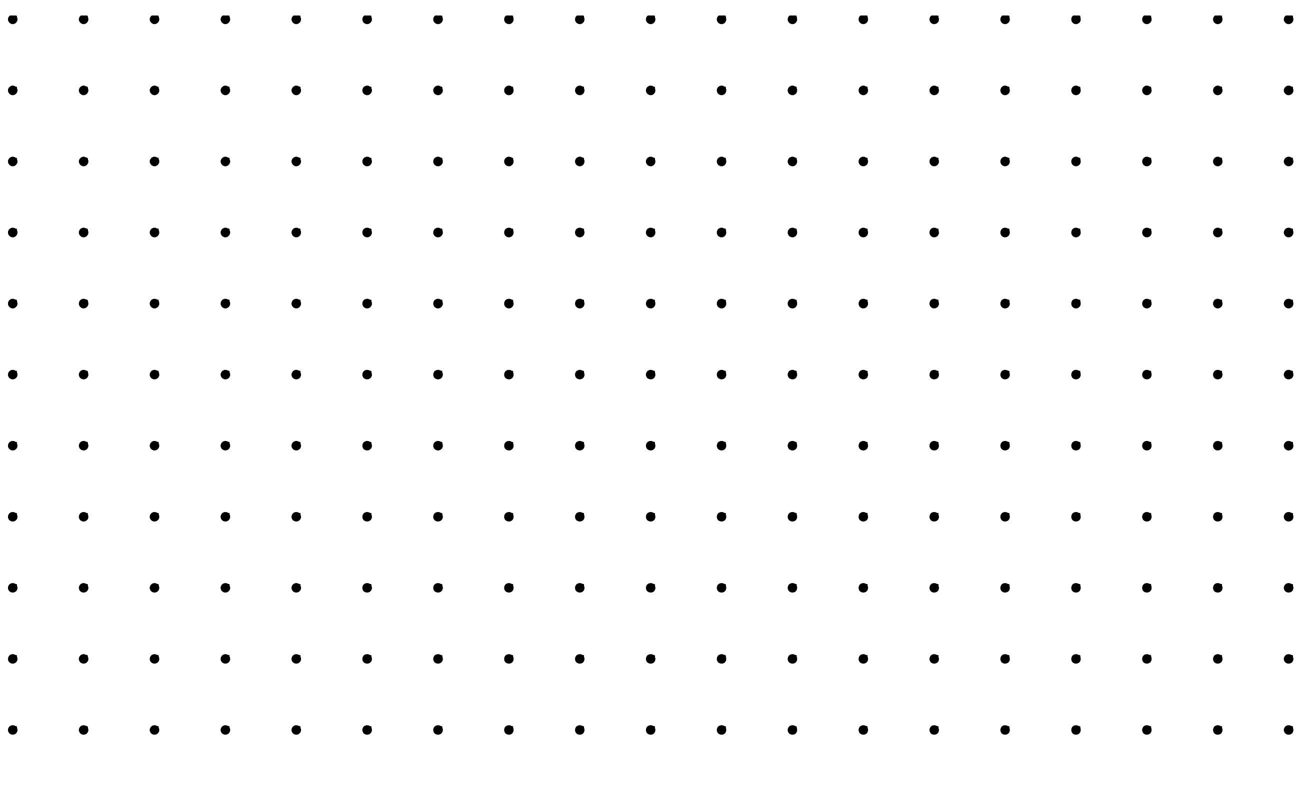
A 2 *by* 3 Square



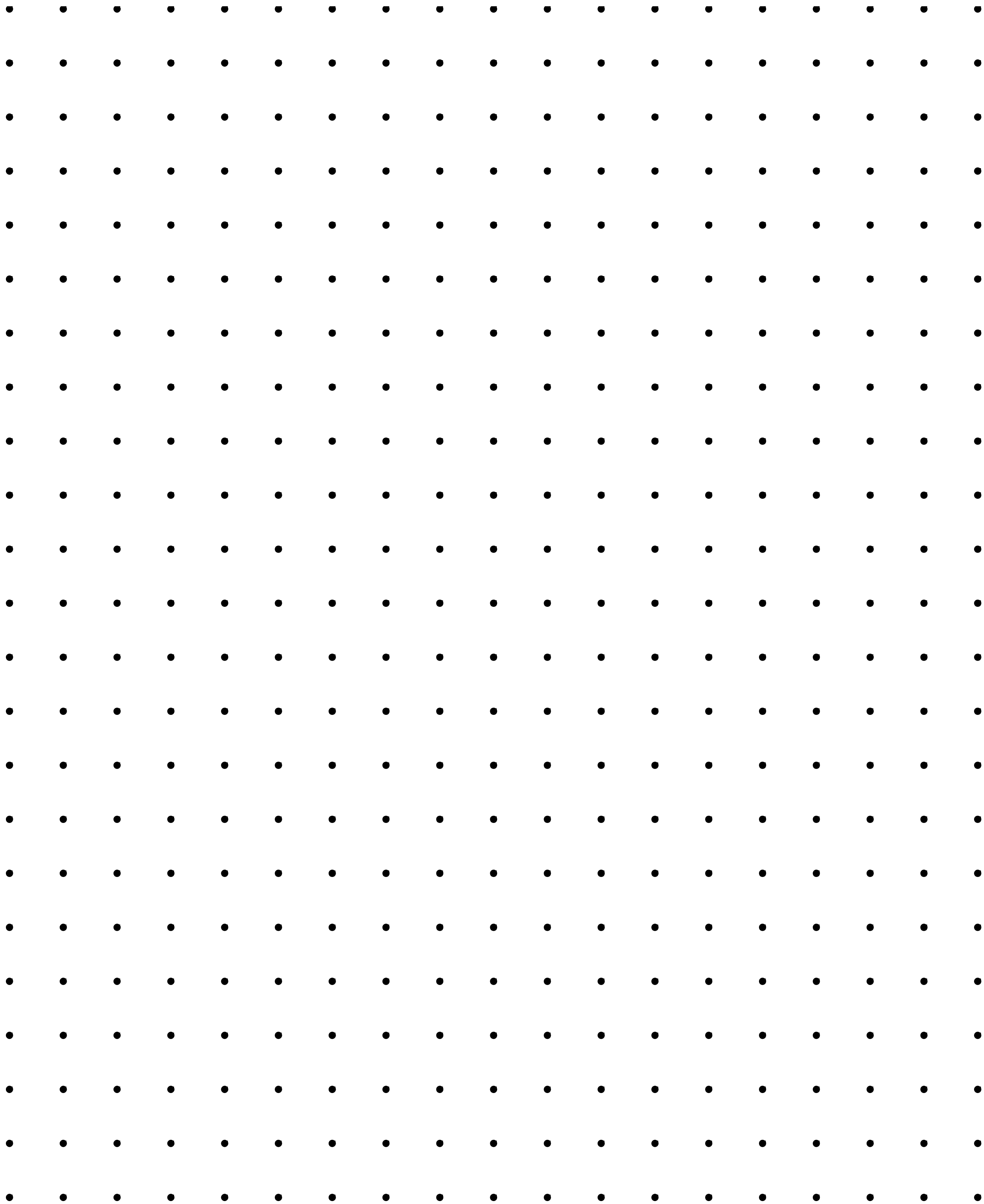
2. In pairs, see if you can find a quick way to figure out the areas of tilted squares.

   What areas can you make by drawing squares on a grid?

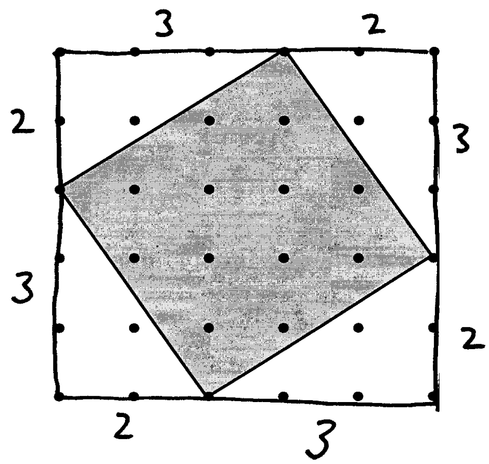
   Remember that you must always join dots to make the squares!



Dotted Grid Paper

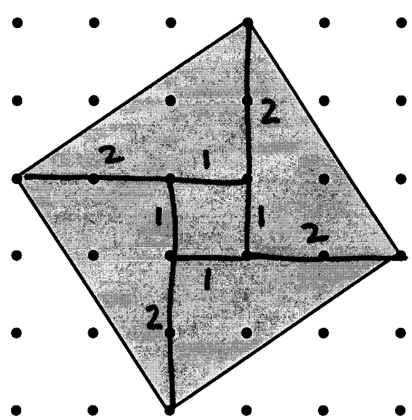


Some Different Approaches



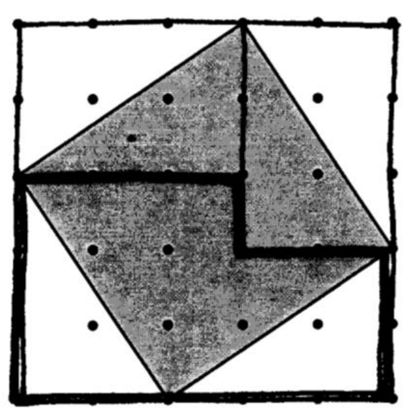
Jason

I drew a square all round the tilted square. I then took away the area of the four right triangles.



Kate

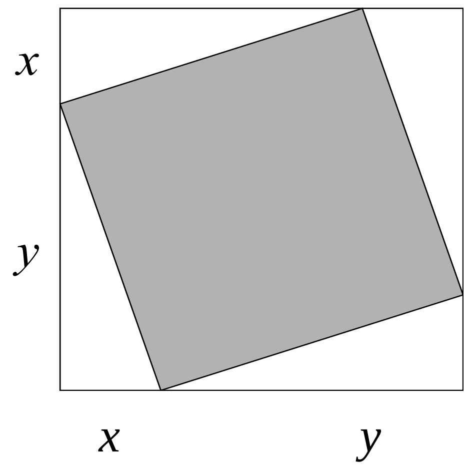
I divided the tilted square into 4 triangles and a little square inside.

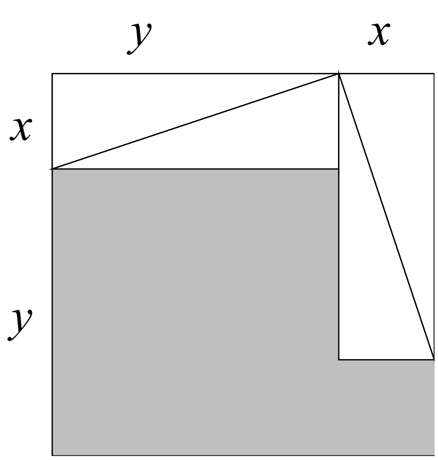


Simon

I found that the area inside the bold line is the same area as the tilted square and used that.

Proving the Pythagorean Theorem





Use what you have learned from the tilted squares problem to do the following:

1. Explain clearly and carefully how you know that the two shaded areas are equal.

2. Write down the shaded areas in terms of the lengths x and y.

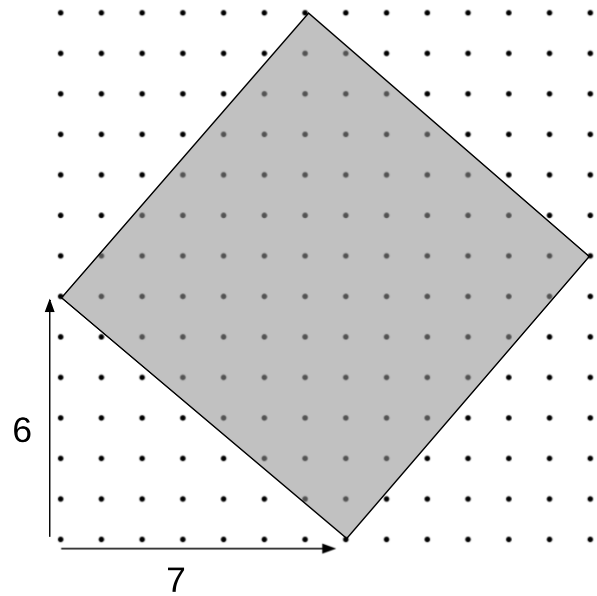
3. If the two shorter sides of a right triangle have lengths x and y, what is the length of the longest side? (This is called the hypotenuse).

Square Areas (revisited)

The dots on the grid are all one unit apart.

1. The square shown here can be described as a 7 *by* 6 square. Find its area.

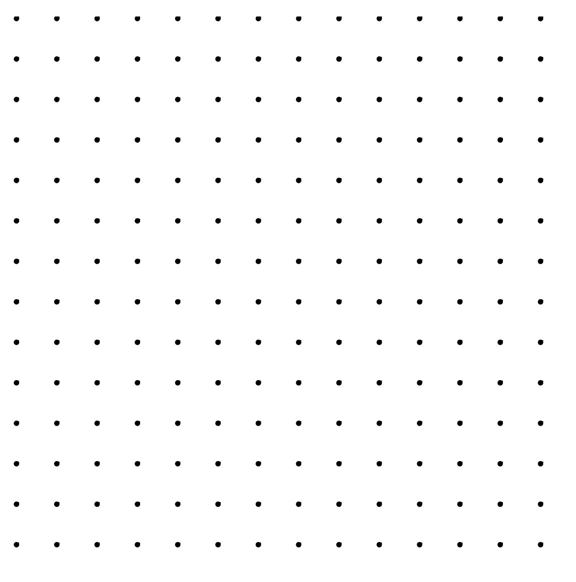
   Show all your reasoning



2. Draw a 7 *by* 5 square.

   Find its area.

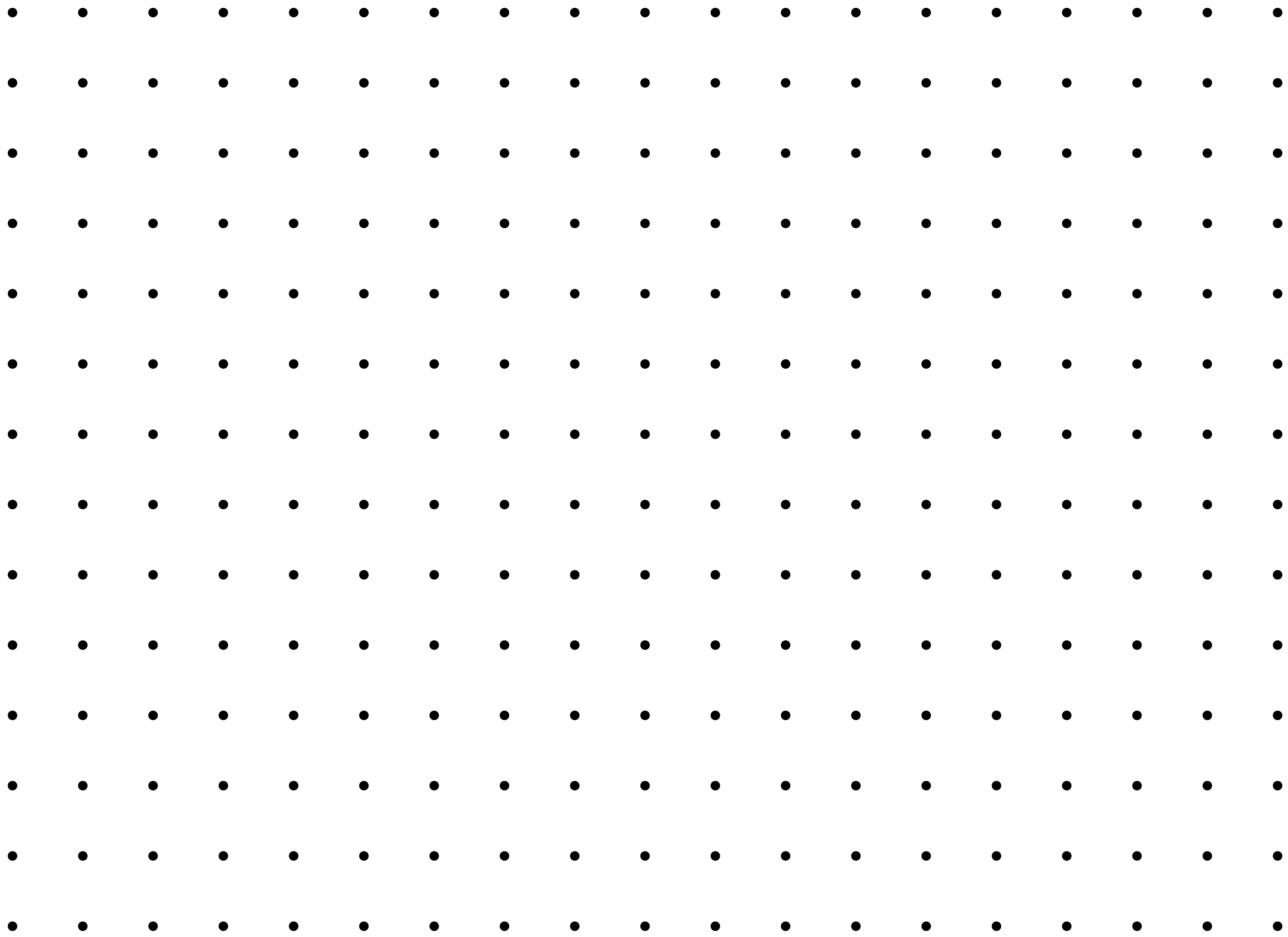
   Show all your reasoning.



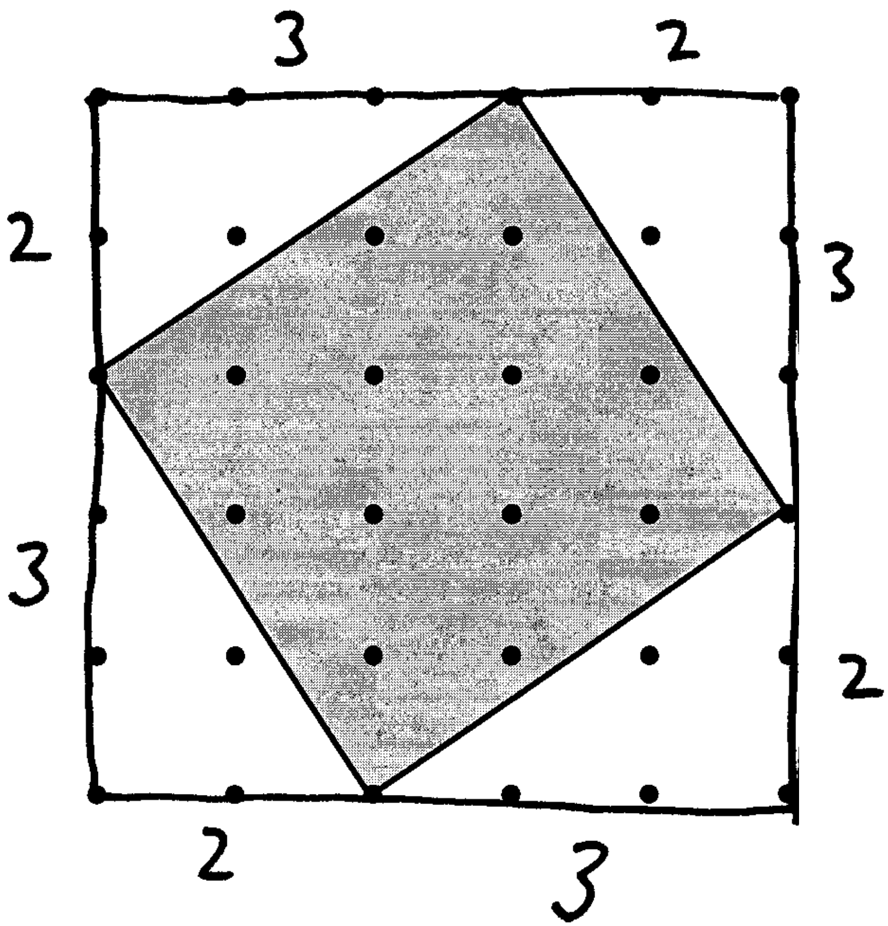
3. Sketch a 7 *by* y square.

   Find its area in terms of y.

   Show all your reasoning.

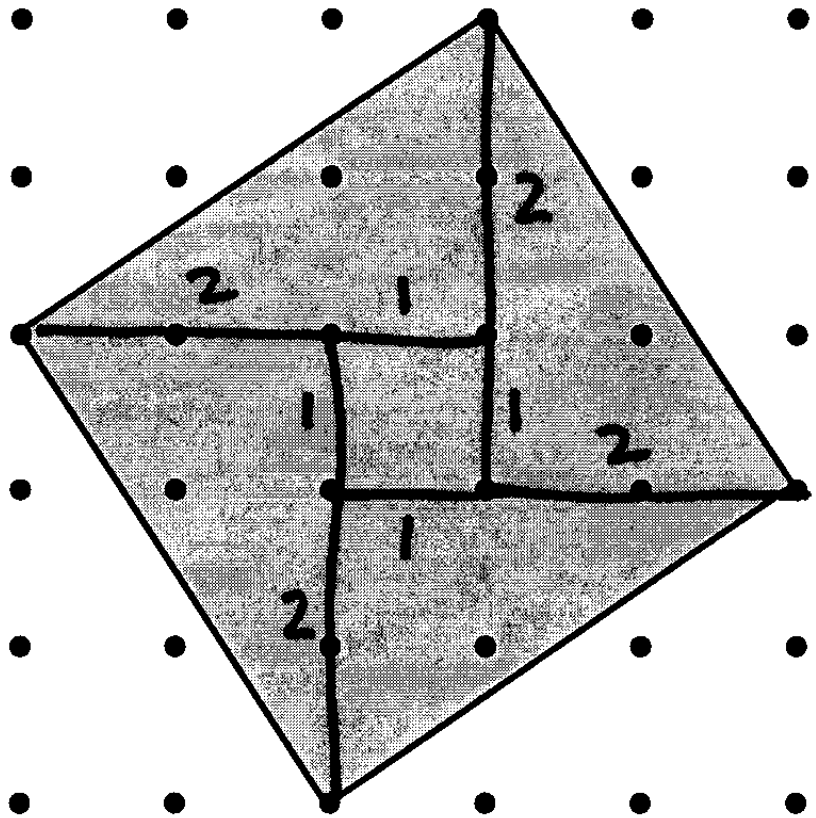


Jason's Method



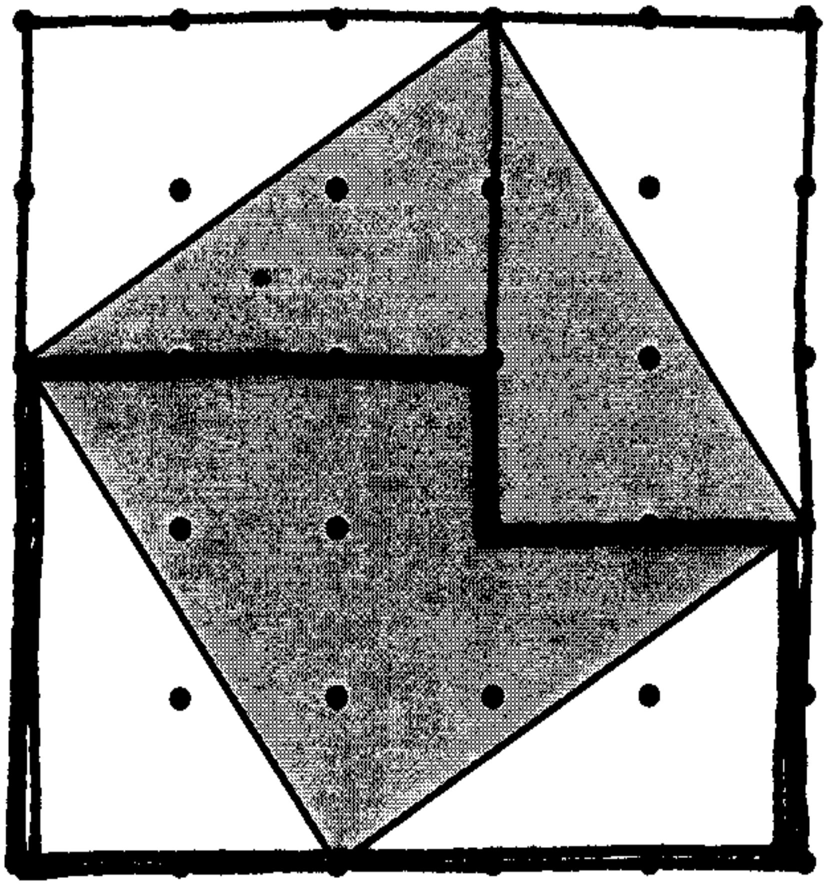
"I drew a square all round the tilted square.
I then took away the area of the four right triangles."

Kate's Method



"I divided the tilted squares into four right triangles and little squares inside."

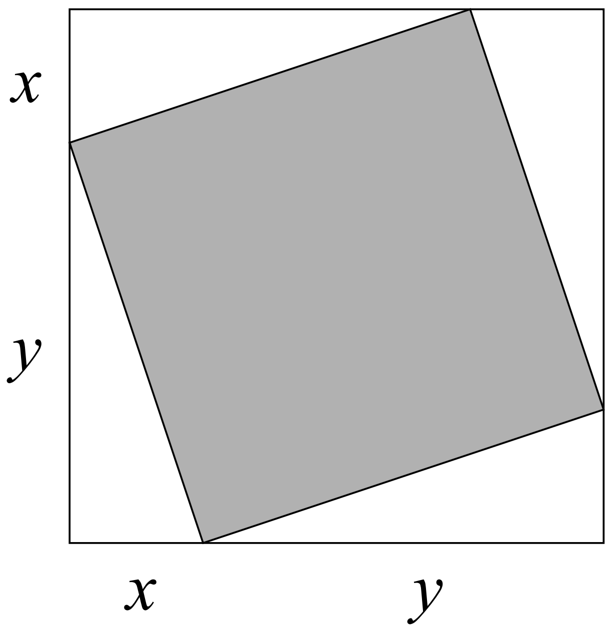
Simon's Method

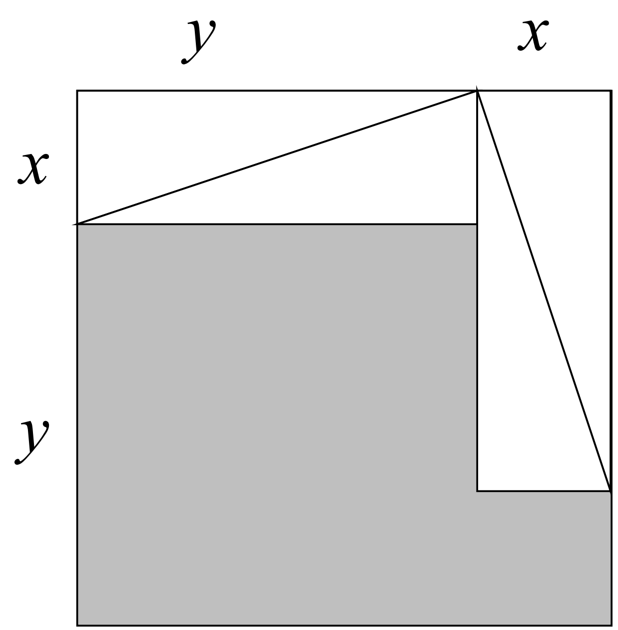


"I found the area inside the bold line is the same area as the tilted square and used that."

x *y*	0	1	2	3	4	Comments
0						
1						
2						
3						
4						

What is the gray area in each case?





Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org