CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Building and Solving Linear Equations

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Building and Solving Linear Equations

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to create and solve linear equations. In particular, the lesson will help you identify and help students who have the following difficulties:

- Solving equations with one variable.
- Solving linear equations in more than one way.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

8.EE: Analyze and solve linear equations and pairs of simultaneous linear equations

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 7 and 8:

1. Make sense of problems and persevere in solving them.
3. Construct viable arguments and critique the reasoning of others.
5. Use appropriate tools strategically.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

The unit is structured in the following way:

- Before the lesson, students work individually on an assessment task that is designed to reveal their current levels of understanding and difficulties. You then review their work and create questions for students to consider when improving their solutions.
- During the lesson students work in pairs on collaborative activities. They evaluate sample student work and create linear equations for each other to solve.
- During a final whole-class discussion students review the main mathematical concepts of the lesson.
- In a follow-up lesson, students review their initial solutions, and then use what they have learned to either revise the same introductory assessment task or complete a different task.

MATERIALS REQUIRED

- Each student will need a copy of the assessment tasks, *Building and Solving Equations, Building and Solving Equations (revisited), Sample Student Work to Discuss*, the cut-up sheet *Building Equations*, the cut-up sheet *Solving Equations*, a sheet of paper, a mini-whiteboard, a pen, and an eraser. Some students may need extra copies of the sheets *Building Equations* and *Solving Equations*.
- There are projector resources to support whole-class discussions.

TIME NEEDED

15 minutes before the lesson, an 80-minute lesson (or two shorter lessons) and 15 minutes in a follow-up lesson. These timings are approximate; exact timings will depend on the needs of your students.

BEFORE THE LESSON

Assessment task: *Building and Solving Equations* (15 minutes)

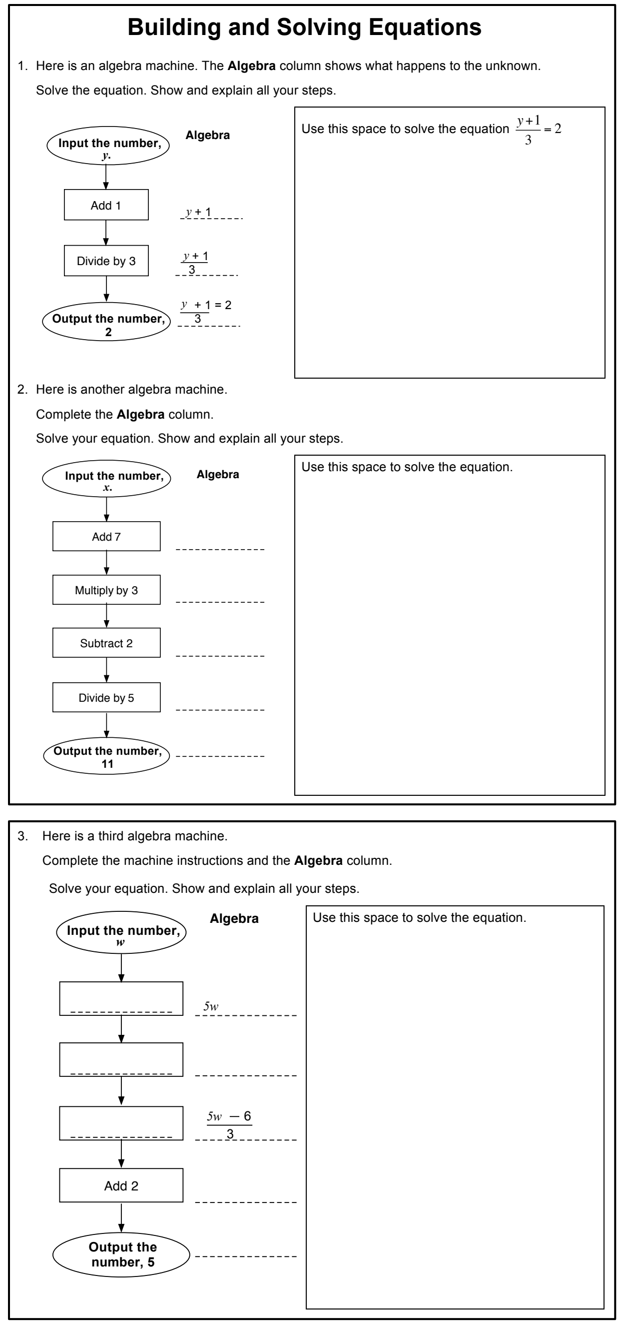
Have the students complete this task, in class or for homework, a few days before the formative assessment lesson. This should give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You will then be able to target your help more effectively in the next lesson.

Give each student a copy of the assessment task *Building and Solving Equations*.

> *Read through the questions and try to answer them as carefully as you can.*

It is important that, as far as possible, students are allowed to answer the questions without your assistance.

Students should not worry too much if they cannot understand or do everything, as in the next lesson, they will engage in a similar task that should help them to progress. Explain to students that by the end of the next lesson, they should expect to answer questions such as these confidently. This is their goal.



Building and Solving Equations

1. Here is an algebra machine. The **Algebra** column shows what happens to the unknown.
   Solve the equation. Show and explain all your steps.

Input the number, y. → Add 1 → Divide by 3 → Output the number, 2

Algebra: $y + 1$; $\frac{y+1}{3}$; $\frac{y+1}{3} = 2$

Use this space to solve the equation $\frac{y+1}{3} = 2$

2. Here is another algebra machine.
   Complete the **Algebra** column.
   Solve your equation. Show and explain all your steps.

Input the number, x. → Add 7 → Multiply by 3 → Subtract 2 → Divide by 5 → Output the number, 11

Algebra

Use this space to solve the equation.

3. Here is a third algebra machine.
   Complete the machine instructions and the **Algebra** column.
   Solve your equation. Show and explain all your steps.

Input the number, w → ____ → ____ → ____ → Add 2 → Output the number, 5

Algebra: $5w$; ____; $\frac{5w - 6}{3}$; ____; ____

Use this space to solve the equation.

Assessing students' responses

Collect students' responses to the task, and note what their work reveals about their current levels of understanding and their individual difficulties.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores, and will distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a list of questions. Some suggestions for these are given in the *Common issues* table on the next page. We suggest that you make a list of your own questions, based on your students' work, using the ideas on the following page. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions, and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students, and write these questions on the board when you return the work to the students.

Common issues:	Suggested questions and prompts:
When solving the equation, different operations are applied to its two sides For example (Q1.): The student writes: $y + 1 = 6$; $y = 7$ (instead of $y = 5$).	• How can you check your solution is correct? • Try substituting 7 for y in each line.
When solving an equation, an expression is incorrectly added to or subtracted from For example: The student attempts to subtract 1 from both sides of the equation (Q1.): $\frac{y+1}{3} = 2$ results in $\frac{y}{3} = 1$.	• Write the expression with the one fraction bar as two fractions. What do you get when you subtract 1? Write the resultant expression on the left side of the equation as one fraction. $\left[\frac{y}{3} + \frac{1}{3}\right] = 2$ • After the subtraction, are the two sides of the equation still equal? How do you know?
When building an equation, just one term in the expression is multiplied For example (Q2.): Multiplying $x + 7$ by 3, the student writes $3x + 7$ instead of $3(x + 7)$, or $3x + 21$.	• Multiply by 3 means multiply the whole expression by 3. How could you write this?
When building an equation, a number is added incorrectly For example: Adding 2 to the expression: $\frac{5w-6}{3}$ the student writes $\frac{5w-4}{3}$ instead of $\frac{5w-6}{3} + 2$.	• Write $\frac{5w-6}{3}$ as two separate terms, both with a denominator 3. Now add 2 to this expression. Compare this with your original expression.
When solving an equation, the distribution law is incorrectly applied For example: $3(x + 7) = 3x + 7$.	• Think of words to describe the expression with parentheses. Do these words match your expression without the parentheses?
When solving an equation, an expression is incorrectly multiplied by a value For example: The student may decide to multiply both sides of the equation by 3 (Q3.): $\frac{5w-6}{3} + 2 = 5$ to obtain $5w - 6 + 2 = 15$.	• Are both sides of your equation equal? How do you know? • Have you multiplied all terms by 3? • Now divide both sides of the equation by 3. Does this give you your original equation?
Student has answered all the questions correctly	• Now use a different method to solve equation 2 or 3.

SUGGESTED LESSON OUTLINE

In this lesson students build and solve complex-looking equations. Understanding how to construct these equations will help students to solve them using a similar step-by-step 'deconstruction' approach.

Throughout the whole-class discussions in this lesson there are some challenging problems; encourage students to first tackle these on their own, and only then discuss them with a neighbor. In that way students will have something to talk about and it may help prevent one student dominating the discussion. Maximize participation in the whole-class discussion by asking all students to show their solutions on their mini-whiteboards. Select a few students with interesting or contrasting answers to justify them to the class. Encourage the rest of the class to contest these explanations.

Whole-class introduction: *Building Equations* (15 minutes)

Give each student a mini-whiteboard, a pen, and an eraser.

In this lesson you will both build equations from solutions such as $x = 2$ and solve equations.

The building of equations will help you understand how to solve them, or 'un-build' them.

Write in the center of the board:

$$x = 6$$

Now x=6 is the solution to the following equation:

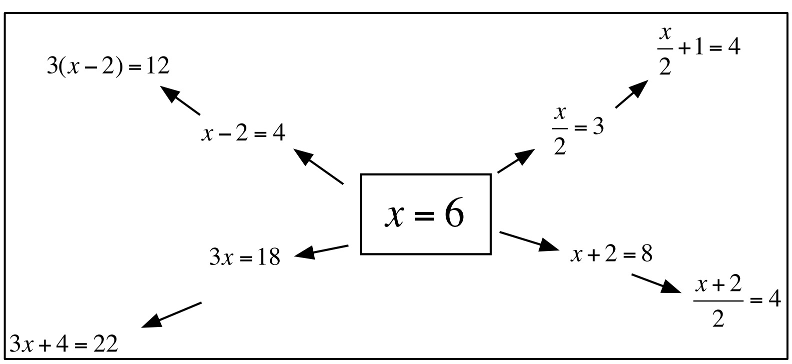
$$x + 3 = 9$$

Now ask the following questions, writing student's answers on the board:

On your mini whiteboard, show me another equation that will give the same solution.
Show me an equation with a subtraction/multiplication/division, where the solution is still $x = 6$.

Can you use one of the equations on the board to build a more complicated one?
Show me an equation with two operations, where the solution is still $x = 6$.

Ask students to show you their mini-whiteboards and record some of their ideas on the board. Show with arrows how more complex equations may be developed from simpler ones, for example:



How can we check that these equations are correct? [Solve them, or substitute $x = 6$ into them.]
Which is the easier method? Why? [Substitution is easier with the more difficult equations.]

There are many methods students may use to create more complex equations. For example, students may use guess and check, or they may create an expression for x to represent the left side of the equation and then substitute $x = 6$ into it to obtain the figure for the right side of the equation. Or they may build up an equation in a step-by-step manner by performing the same operation on both sides of the equation; for instance, they may start by multiplying both sides of the equation by 5 to produce the equation $5x = 30$. At this stage it does not matter if students have not found a general method to build an equation, as the next activity will help.

Collaborative activity: *Sample Student Work* (10 minutes)

Organize the class into pairs.

Give each pair the *Sample Student Work to Discuss*, and ask for written comments.

In pairs you are going to see how two students attempted to build an equation. They started with $x = 6$ and made this more complicated, step-by-step. They did not do this perfectly and your job is to improve their work.

Take turns to work through a student's work.
Explain the work to your partner.

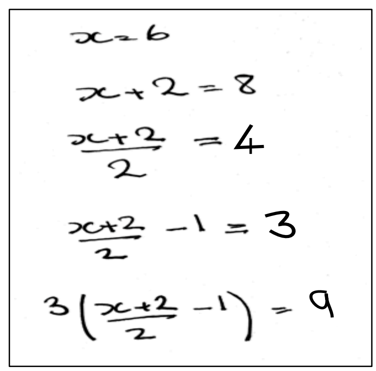
Listen carefully to explanations. Ask questions if you don't understand.

Once you are both satisfied with the explanations, correct any mistakes and improve the work.

Slide P-1 of the projector resource, *Evaluating Sample Student Responses*, describes how students should work together. During the group work, observe, listen, and support the students. Note any difficulties students have with understanding the work. Check, also, to find any of the responses students have difficulties in understanding. This information can help you focus the next activity, a whole-class discussion.

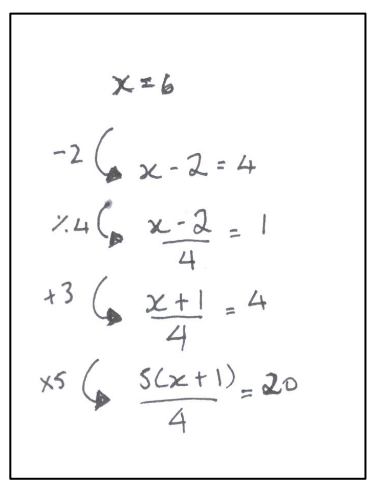
Both pieces of sample student work use similar approaches:

Amy has correctly built an equation and has not made any mistakes. She has not explained each stage in the construction. When students need your support during the discussions, ask them if they can figure out what Amy did at each step and if they can think of a way of improving her presentation [e.g. using connecting arrows, like Laura, below].



Laura has used arrows to connect the steps she has taken and she has shown at each step what she has done (or intended to do). However, she made a mistake when building her equation. When intending to add 3 to both sides of the equation she added 3/4 to the expression on the left side of the equation.

If students do not spot Laura's mistake, ask them to substitute 6 into each step of the equation. Ask them if the equation holds true at each step. This will give them a useful strategy for checking their own equations.



Whole-class Discussion: *Deconstructing and Solving Equations* (15 minutes)

Organize a brief whole-class discussion to evaluate the different pieces of work.

You can display the sample student work using Slides P-2 and P-3 of the projector resource.

Ask the following questions in turn:

Can you explain how the students have built their equation?

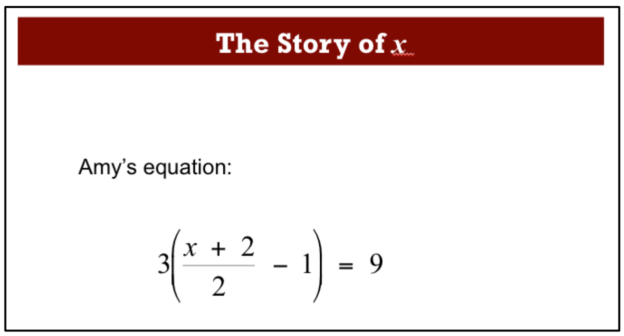
How does this method compare to the one you used to build an equation?
Could this method be applied to the construction of any equation?

Can you think of ways the students could improve the presentation of their work?

Has either student made a mistake?

How can you check their work? [Substitution of x = 6 into the equation, or solving the equation.]
Which method of checking is easier for harder equations? [Substitution.]

Re-introduce Amy's equation by showing Slide P-4, *The Story of x*, of the projector resource.



Equations can tell the story of 'a day in the life of x.'

Look at Amy's equation. When it was being built, can you tell what Amy did first? [Added 2.]

What did she do last? [Multiplied by 3.]

Just by looking at the equation can you tell the story of the construction of Amy's equation? Describe it to me.

So if we were to solve Amy's equation, what would we do first? [Divide by 3].

Then what would we do? [Add 1.] ...

Ask students to solve Amy's equation on their mini white-boards by 'deconstructing' it, in a step-by-step manner, without missing out any steps. Just as Laura, in the sample work, described each step when building the equation, students are to describe each step when solving the equations, using arrows to show the steps.

Ask students to demonstrate their method on the board.

The solution to Amy's equation should look like this (add arrows if students have not done so):

$$3\left(\frac{x+2}{2}-1\right)=9$$

Divide by 3

$$\frac{x+2}{2}-1=3$$

Add 1

$$\frac{x+2}{2}=4$$

Multiply by 2

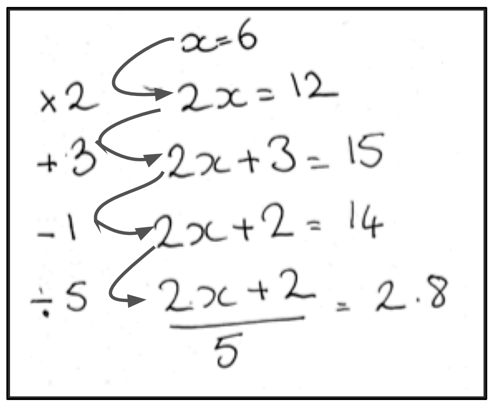
$$x+2=8$$

Subtract 2

$$x=6$$

Compare your method with the one Amy used to build up an equation. What do you notice?
[When solving Amy's equation use the inverse operations to those Amy used when building the equation, in reverse order.]

Show students the following sample of work produced by Rob (it can be found on Slide P-5). In step 2 Rob has added 3 and in step 3 he has subtracted 1. From the final equation, however, you cannot tell that this was done.



Point out that solving an equation need not always be the exact reverse of the operations used to make it. (In this case it would be ×5, -2, ÷2.)

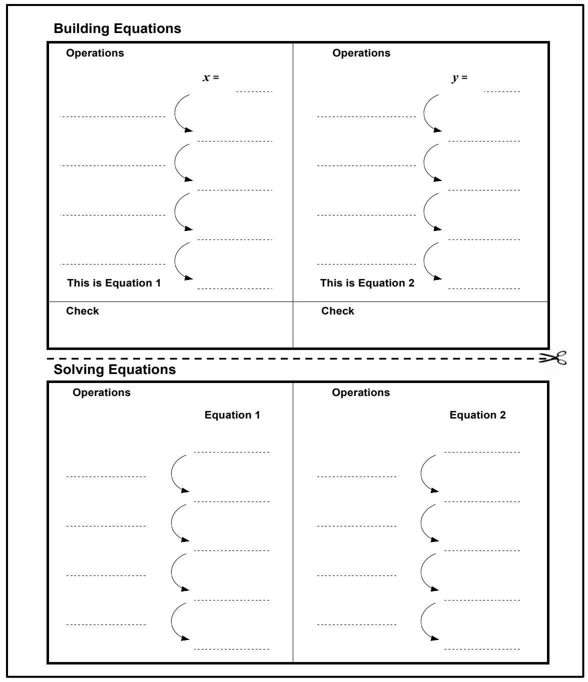
It is only when terms in an equation are not manipulated that the operations required to solve the same equation are the exact reverse of those used to build it.

Explain to students that the purpose of the lesson is to build up quite complicated equations and then ask a partner to solve them. Understanding the process of constructing an equation will help students solve them by a process of 'deconstruction'.

Extending the lesson over two days

If you are taking two days to complete this lesson unit then you may want to end the first lesson here. At the start of the second day, briefly remind students of the task they have been working on before moving on to the collaborative activity.

Collaborative activity: *Building and Solving Equations* (25 minutes)



Give each pair of students the sheets *Building Equations* and *Solving Equations*.

> *Your task now is to create two equations for your partner to solve and then to solve your partner's equations.*

Use Slides P-6 and P-7, *Working Together*, of the projector resource as you explain what to do:

First task: Building and checking equations

> *Your first task is to build and check two equations. Use your own value for x!*
> *Use the sheet* ***Building Equations****, to do this.*
>
> *Each equation should use all four operations +, −, ×, and ÷ and four different integers.*
>
> *Make sure the order of the operations is different for each equation.*
>
> *Check that each step works by substituting your original value for x into it.*
>
> *Then write your finished two equations at the top of the sheet* ***Solving Equations****.*

Second task: Solving equations

> *Your second task is to solve each other's equations.*
>
> *Give your sheet to your partner and ask them to solve the equations.*
> *Help your partner if they become stuck.*
>
> *If your partner's answers are different from yours, ask for an explanation. If you still don't agree, explain your own thinking.*
>
> *It is important that you both agree on the answers.*
>
> *Once you've established the correct answer, check to see if the steps used to solve the equation are the reverse of the steps used to build the equation. If they are not, try to figure out why.*

The purpose of this structured group work is to encourage students to engage with each other's explanations, and take responsibility for each other's understanding.

While students are working in small groups you have two tasks: to note how students approach the task, and to support student reasoning.

Note different student approaches to the task

Notice how students make a start on the task, where they get stuck, and how they respond if they do come to a halt. Notice any errors. Students may make calculation errors when substituting into equations. They may use non-standard notation. Students may forget to perform operations on both sides of the equation, or not use the distribution property accurately when solving equations. Students may multiply or divide just one term in the expression, instead of the whole expression on both sides of the equation.

You can use this information to focus the whole-class discussion towards the end of the lesson.

Support student reasoning

Try not to make suggestions that resolve errors and difficulties for students. Instead, ask questions to help students to reason together to identify and resolve issues.

Did your partner find the value of x you started with?

Can you explain what you wrote here?

How do you 'divide the whole of the expression on the left-hand side by 3'?

Explain how you know which operation to undo first.

Can you find a different way of writing this expression?

How do you know these two expressions are equal?

Are the operations you used to solve the equation the exact reverse of those used to build it? If not, why not? [This could be due to an algebraic error made when building or solving, or perhaps the builder has simplified an expression in the equation.]

What could you change so that the two sets of operations are the exact reverse of each other? [When building the equation do not simplify any of the expressions.]

The questions in the *Common issues* table may also help you support your students.

Encourage students who quickly complete the two sheets to create more challenging equations without using the structured sheets. Students could build equations with parentheses or equations that include $\frac{1}{x}$.

Whole-class discussion (15 minutes)

Organize a discussion about what has been learned. Depending on how the lesson went, you may want to focus on the common mistakes made by students, review what has been learnt, or you may want to extend and generalize the math.

Throughout this discussion encourage students to justify their answers. Try not to correct answers, instead encourage students to challenge each other's explanations.

Write this equation on the board:

$$\frac{3(2a-1)+2}{7}=5$$

Show me a method for solving this equation.

After a few minutes ask students to show you their whiteboards. Ask two or three students with different answers to justify them to the rest of the class.

This is one possible method:

$$\frac{3(2a - 1) + 2}{7} = 5$$

Multiply by 7

$$3(2a - 1) + 2 = 35$$

Subtract 2

$$3(2a - 1) = 33$$

Divide by 3

$$2a - 1 = 11$$

Add 1

$$2a = 12$$

Divide by 2

$$a = 6$$

Ask students to critique each other's solution methods.

Does anyone disagree with this method?

Does anyone have a different method?

Does anyone have a more efficient method?

Follow-up lesson: reviewing the assessment task (15 minutes)

Return the original assessment *Building and Solving Equations* to the students.

If you have not added questions to individual pieces of work, then write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

Read through your original solutions to the task and think about what you have learned this lesson.

Carefully answer the questions I have written.

If students struggled with the original assessment then they may benefit from revising this assessment. In order that students can see their own progress, ask them to complete the task using a different color pen or give them a second blank copy of the task. Otherwise give students a copy of the task *Building and Solving Equations (revisited)*.

Use what you have learned to complete the new assessment task/revise your answers.

You could give this task for homework.

SOLUTIONS

Assessment Task: *Building and Solving Equations*

There are several ways to solve each equation. Below are some examples.

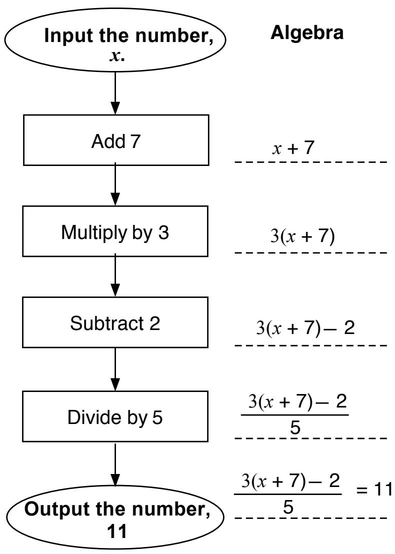
1.

$$\frac{y+1}{3} = 2$$

multiply by 3 $\quad y + 1 = 6$

subtract 1 $\quad y = 5.$

2.



$$\frac{3(x+7)-2}{5} = 11$$

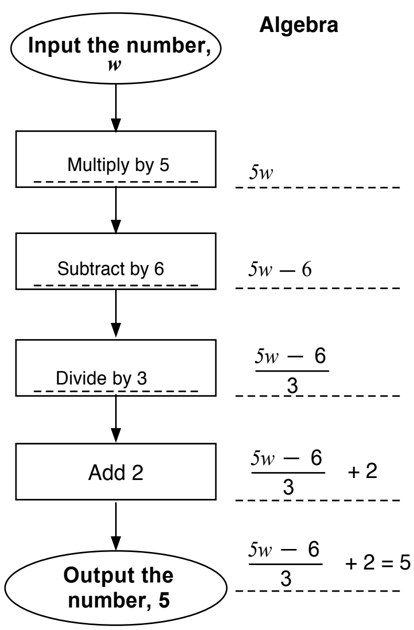
multiply by 5 $\quad 3(x+7) - 2 = 55$

add 2 $\quad 3(x+7) = 57$

divide by 3 $\quad x + 7 = 19$

subtract 7 $\quad x = 12.$

3.



$$\frac{5w-6}{3} + 2 = 5$$

subtract 2 $\quad \frac{5w-6}{3} = 3$

multiply by 3 $\quad 5w - 6 = 9$

add 6 $\quad 5w = 15$

divide by 5 $\quad w = 3.$

Assessment Task: *Building and Solving Equations (revisited)*

There are several ways to solve each equation. Below are some examples.

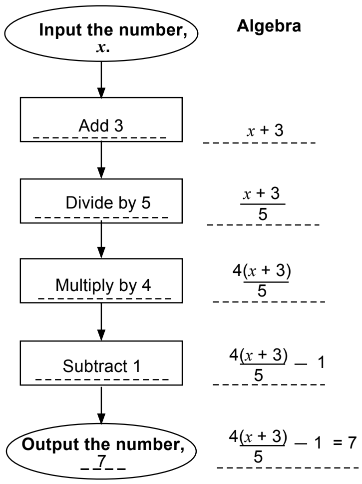
1.

$$5(y+2) = 25$$

divide by 5 $\quad y + 2 = 5$

subtract 2 $\quad y = 3.$

2.



$$\frac{4(x+3)}{5} - 1 = 7$$

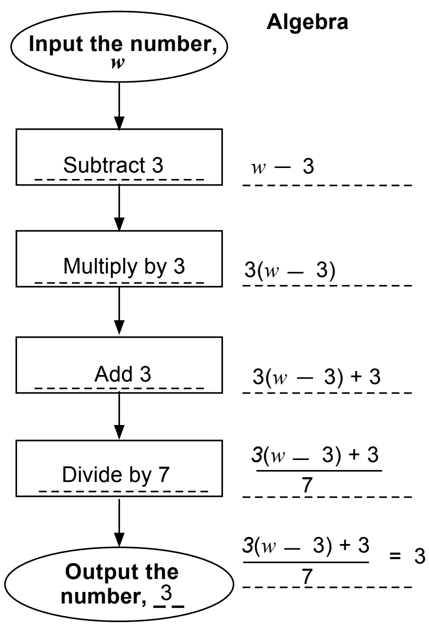
add 1 $\quad \frac{4(x+3)}{5} = 8$

multiply by 5 $\quad 4(x+3) = 40$

divide by 4 $\quad x + 3 = 10$

subtract 3 $\quad x = 7.$

3.



$$\frac{3(w-3)+3}{7} = 3$$

multiply by 7 $\quad 3(w-3) + 3 = 21$

subtract 3 $\quad 3(w-3) = 18$

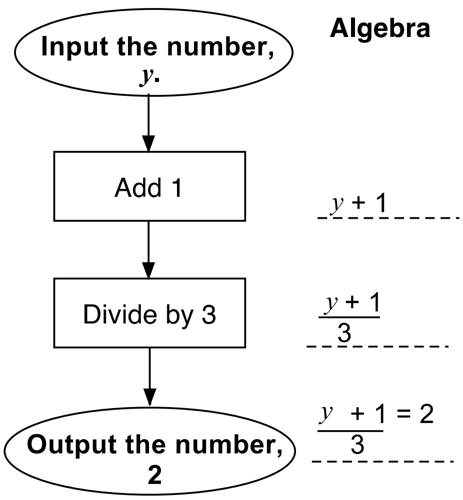
divide by 3 $\quad w - 3 = 6$

add 3 $\quad w = 9.$

Building and Solving Linear Equations

1. Here is an algebra machine. The **Algebra** column shows what happens to the unknown.

   Solve the equation. Show and explain all your steps.

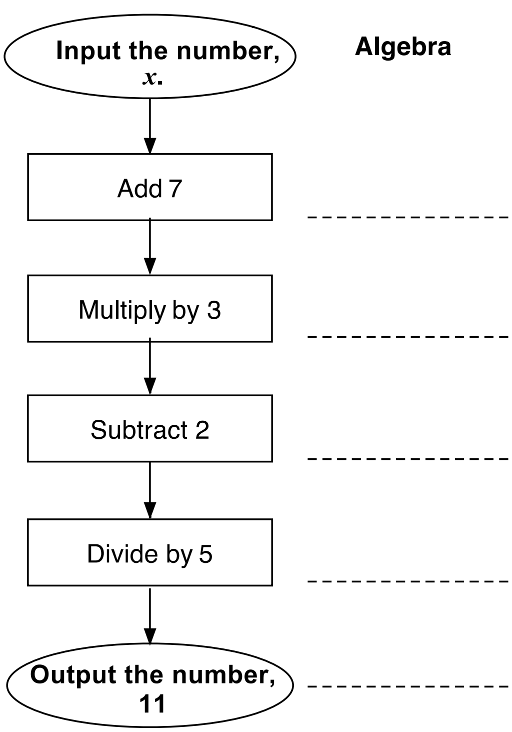


Use this space to solve the equation $\frac{y+1}{3} = 2$

2. Here is another algebra machine.

   Complete the **Algebra** column.

   Solve your equation. Show and explain all your steps.

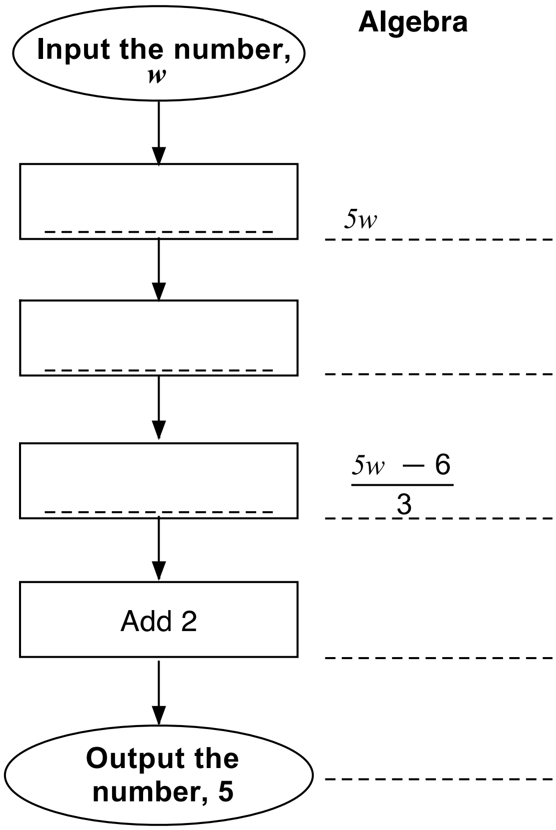


Use this space to solve the equation.

3. Here is a third algebra machine.

Complete the machine instructions and the **Algebra** column.

Solve your equation. Show and explain all your steps.



Use this space to solve the equation.

Sample Student Work to Discuss

Correct any mistakes and improve the work.

Amy:

$$x = 6$$

$$x + 2 = 8$$

$$\frac{x+2}{2} = 4$$

$$\frac{x+2}{2} - 1 = 3$$

$$3\left(\frac{x+2}{2} - 1\right) = 9$$

Laura:

$$x = 6$$

-2 $$x - 2 = 4$$

$\div 4$ $$\frac{x-2}{4} = 1$$

$+3$ $$\frac{x+1}{4} = 4$$

$\times 5$ $$\frac{5(x+1)}{4} = 20$$

Building Equations

Operations	Operations
$x =$	$y =$
This is Equation 1	This is Equation 2
Check	Check

Solving Equations

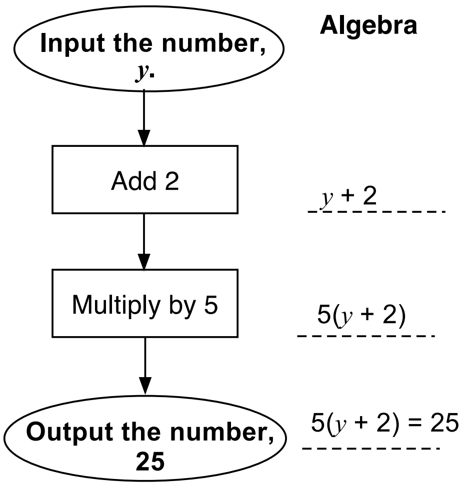
Operations	Operations
Equation 1	Equation 2

Building and Solving Linear Equations (revisited)

1. Here is an algebra machine.

   The **Algebra** column shows what happens to the unknown.

   Solve the equation. Show and explain all your steps.

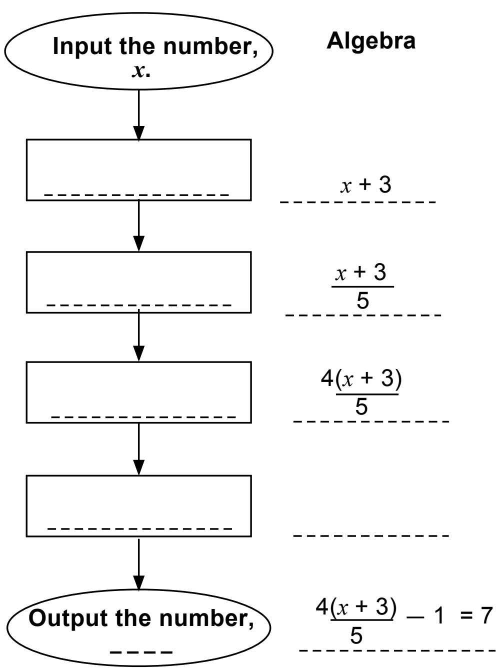


Use this space to solve the equation: $5(y+2)=25$

2. Here is another algebra machine.

   Complete the machine instructions and the **Algebra** column.

   Solve your equation. Show and explain all your steps.

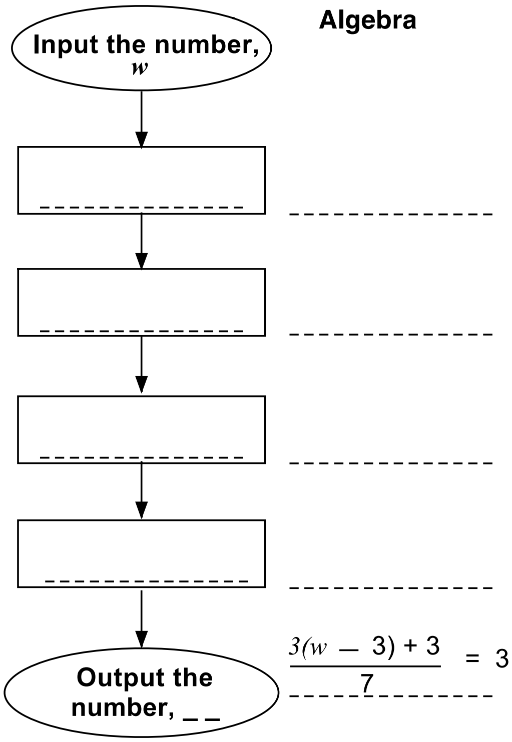


Use this space to solve the equation: $\frac{4(x+3)}{5}-1=7$

3. Here is a third algebra machine.

Complete the machine instructions and the **Algebra** column.

Solve your equation. Show and explain all your steps.



Use this space to solve the equation: $\frac{3(w-3)+3}{7}=3$

Evaluating Student Responses

1. Take turns to work through a student's work.
   Explain the work to your partner.

2. Listen carefully to explanations.
   Ask questions if you don't understand.

3. Once you are both satisfied with the explanations, correct mistakes and make any improvements to the work.

Sample Student Work: Amy

$$x = 6$$

$$x + 2 = 8$$

$$\frac{x+2}{2} = 4$$

$$\frac{x+2}{2} - 1 = 3$$

$$3\left(\frac{x+2}{2} - 1\right) = 9$$

Sample Student Work: Laura

$x = 6$

-2 ↳ $x - 2 = 4$

$\div 4$ ↳ $\frac{x-2}{4} = 1$

$+3$ ↳ $\frac{x+1}{4} = 4$

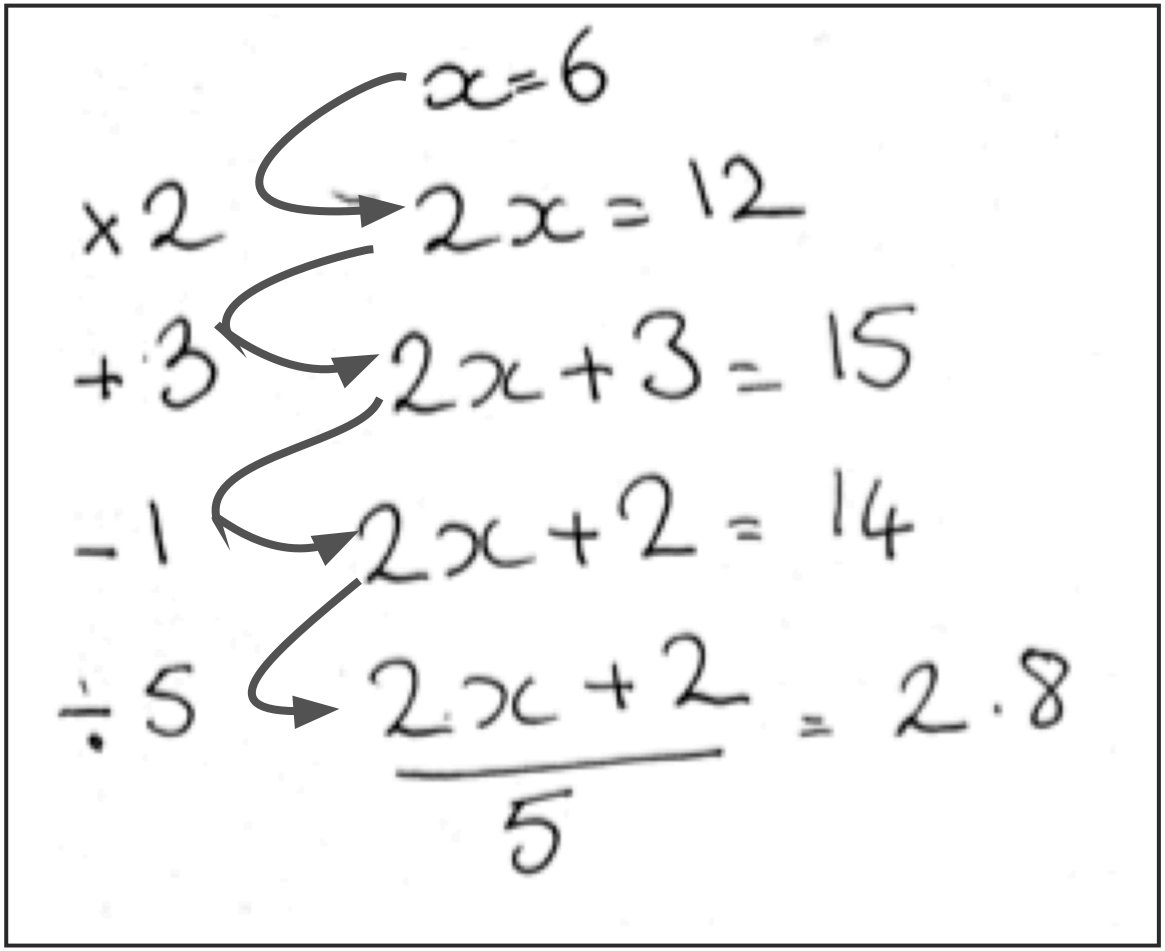
$\times 5$ ↳ $\frac{5(x+1)}{4} = 20$

The Story of x

Amy's equation:

$$3\left(\frac{x+2}{2}-1\right)=9$$

Sample Student Work: Rob



Working Together: Building and Checking Equations

1. Make up your own value for *x*!
2. To build an equation use each of the four operations:
   +, −, ×, and ÷ and four different integers.
   Make sure the order of the operations is different for each equation.
3. Use substitution to check that each step is correct.
4. Write your finished equations at the top of the sheet *Solving Equations*.

Working Together: Solving Equations

1. Ask your partner to solve your two equations. Help your partner if they get stuck.
2. If your partner's answers are different from yours, ask for an explanation. If you still don't agree, explain your own thinking. It is important that you both agree on the answers.
3. Check also to see if the steps used to solve the equation are the reverse of the steps used to build the equation. If they are not, try to figure out why.

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org