CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Classifying Solutions to Systems of Equations

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

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Classifying Solutions to Systems of Equations

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Classify solutions to a pair of linear equations by considering their graphical representations.
- Use substitution to complete a table of values for a linear equation.
- Identify a linear equation from a given table of values.
- Graph and solve linear equations.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

8.EE: Analyze and solve linear equations and pairs of simultaneous linear equations.

This lesson also relates to **all** the *Standards for Mathematical Practice*, with a particular emphasis on:

7. Look for and make use of structure.

INTRODUCTION

- Before the lesson, students attempt the assessment task individually. You then review students' solutions and formulate questions that will help them improve their work.
- During the lesson, students work collaboratively in pairs or threes, plotting graphs, completing tables of values and deducing equations. Then, based on the number of common solutions, students link these representations.
- In a follow-up lesson, students receive your comments on the assessment task and use these to attempt the similar task, approaching it with insights gained from the lesson.

MATERIALS REQUIRED

- Each student will need a copy of the assessment tasks *Working with Linear Equations* and *Working with Linear Equations (revisited),* a mini-whiteboard, eraser, and a pen.
- Each group of students will need cut up *Card Set A*: *Equations, Tables & Graphs*, two cut up copies of *Card Set B*: *Arrows*, one copy of *Graph Transparency*, copied onto a transparency, a transparency pen, a large sheet of paper for making a poster, some plain paper, and a glue stick.
- Provide rulers if requested. There are some projector resources to support whole-class discussion.

TIME NEEDED

15 minutes before the lesson for the assessment task, a 80-minute lesson (or split into two shorter lessons), and 15 minutes in a follow-up lesson (or for homework). All timings are approximate.

BEFORE THE LESSON

Assessment task: *Working with Linear Equations* (15 minutes)

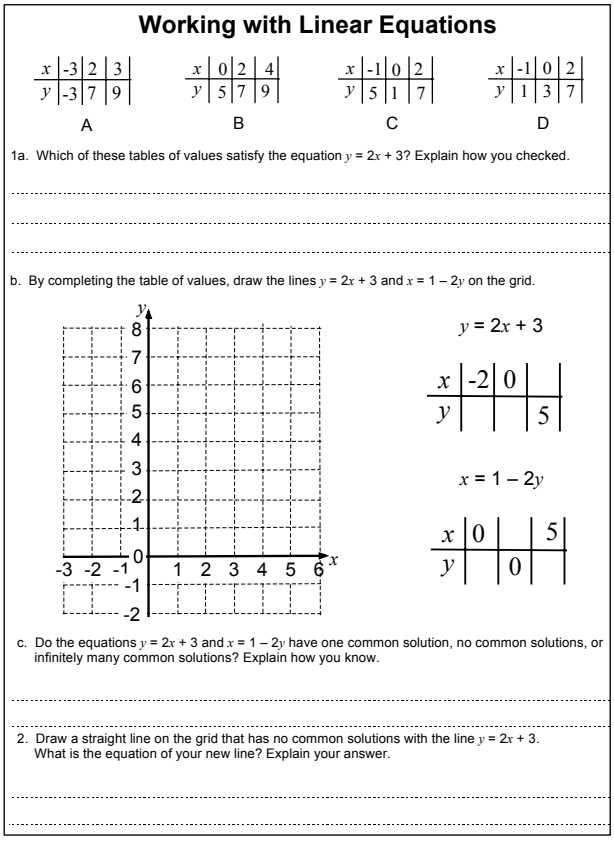
Ask the students to do this task, in class or for homework, a day or more before the formative assessment lesson. This will give you an opportunity to assess the work and identify students who have misconceptions or need other forms of help. You will then be able to target your help more effectively in the next lesson.

Give each student a copy of *Working with Linear Equations*.

Introduce the task briefly, and help the class to understand what they are being asked to do. You may want to explain to the class the term 'common solution'.

> *Spend fifteen minutes working individually, answering these questions.*
>
> *Show all your work on the sheet.*
>
> *Make sure you explain your answers really clearly.*

It is important that students answer the questions without assistance, as far as possible.



Working with Linear Equations

x	-3	2	3
y	-3	7	9

A

x	0	2	4
y	5	7	9

B

x	-1	0	2
y	5	1	7

C

x	-1	0	2
y	1	3	7

D

1a. Which of these tables of values satisfy the equation $y = 2x + 3$? Explain how you checked.

b. By completing the table of values, draw the lines $y = 2x + 3$ and $x = 1 - 2y$ on the grid.

$y = 2x + 3$

x	-2	0	
y			5

$x = 1 - 2y$

x	0		5
y		0	

c. Do the equations $y = 2x + 3$ and $x = 1 - 2y$ have one common solution, no common solutions, or infinitely many common solutions? Explain how you know.

2. Draw a straight line on the grid that has no common solutions with the line $y = 2x + 3$. What is the equation of your new line? Explain your answer.

Students should not worry too much if they cannot understand or do everything because you will teach a lesson using a similar task, which should help them. Explain to students that by the end of the next lesson, they should expect to answer questions such as these. This is their goal.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and difficulties. The purpose of this is to forewarn you of the issues that will arise during the lesson, so that you can prepare carefully.

We suggest that you do not score students' work. The research shows that this is counterproductive, as it encourages students to compare scores and distracts their attention from how they may improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a list of questions. Some suggestions for these are given in the *Common issues* table on the next page. We suggest that you make a list of your own questions, based on your students' work, using the ideas in the table. We recommend you write one or two questions on each student's work. If you do not have time to do this, you could:

- Select a few questions that will be of help to the majority of students, and write these questions on the board when you return the work to the students.
- Or give each student a printed version of your list of questions and highlight the questions for each individual student.

Common issues:	Suggested questions and prompts:
Student assumes that only one of the tables satisfies the equation $y = 2x + 3$ (Q1) For example: The student selects only table A.	• Are there more than three pairs of values that satisfy the equation y = $2x + 3$? • Have you checked the values for x and y in each of the tables?
Student makes an incorrect assumption about the multiplicative properties of zero (Q1 & Q2) For example: The student assumes $2 \times 0 + 3 = 5$. They then may select Table B as satisfying the equation $y = 2x + 3$ (Q1)	• Is 4×0 the same as 4×1? • Use addition to figure out two multiplied by zero. [E.g. $0 + 0 = 0$.]
Student applies the rules for multiplying negative numbers incorrectly (Q1 & Q2) For example: The student assumes $2 \times -1 + 3 = 5$ They then may select Table C as satisfying the equation $y = 2x + 3$ (Q1)	• Is 3×-2 the same as 3×2?
Student provides little or no explanation (Q1)	• What method did you use when checking which tables satisfy the equation? Write what you did.
Student incorrectly draws the graph For example: The student draws a non-linear graph.	• On your graph, is the slope always the same? Does this agree with the equation of the graph? • How can you check you have plotted the graph correctly?
Student uses guess and check to complete the tables of values (Q1b)	• Can you think of a quicker method? • Would changing the subject of the equation help you figure out some of the values?
Student states that the two equations, $y = 2x + 3$ and $x = 1 - 2y$ have no common solutions (Q1c) For example: The student fails to extend the line $x = 1 - 2y$ beyond the values in the table. This means the two lines do not intersect.	• What does 'common solution' mean? • Are there any other points that satisfy the equation $x = 1 - 2y$? Plot some.
Student provides little or no explanation (1c and 2)	• Explain why you think your answer is correct.
Student either does not plot a line that has no common solutions with the line $y = 2x + 3$ or plots it incorrectly (Q2)	• Sketch two lines that have no common solutions. What property do they share? [The lines will be parallel.]
Student uses guess and check to figure out the equation of the line (Q2)	• Can you think of a quicker method? • What can you tell me about two lines with no common solution? Give me two equations that have no common solution.

SUGGESTED LESSON OUTLINE

Whole-class introduction (10 minutes)

Give each student a mini-whiteboard, pen, and eraser. Maximize participation in the discussion by asking all students to show you solutions on their mini-whiteboards.

Write the equation $y = 3x + 2$ on the board.

Ask the following questions in turn:

If $x = 5$ what does y equal? [17]

Ask students to explain how they arrived at their answer. If a variety of values are given within the class, discuss any common mistakes and explore different strategies.

If $x = -1$ what does y equal? [-1]

If students are struggling with multiplying by a negative number, ask the class to summarize the results of multiplying with positives and negatives. Some students may believe that because x and y are different letters, they have to take different values. Point out that here both x and y can both be equal to -1.

If $y = 8$ what does x equal? [2]

If $y = 0$ what does x equal? [$-\frac{2}{3}$]

Students may either use guess and check or rearrange the equation in order to figure out the value for x. You may want to discuss these two strategies.

Students often think that they have made a mistake when they get an answer that is not a whole number. Discuss the value of checking an answer by substituting it back in as x, as well as emphasizing that not all solutions will be positive integers and that negative and fractional solutions can also occur.

It may also be appropriate to discuss the benefits of leaving answers in fraction form rather than converting to a decimal, especially when a recurring decimal will result. Provide an example of say, ½, and discuss the difference between this fraction expressed as a decimal, and $-\frac{2}{3}$ expressed as a decimal, in terms of accuracy and rounding.

How can you check your answer? [By substituting it back in as x.]

How can you check that all your answers are correct? [Sketch the coordinates on a grid and see if they form a straight line.]

If students' work on the assessment task has highlighted issues with plotting points and making connections between solutions to a linear equation and points on a straight line graph, it may be appropriate to ask students to check that the solutions for the equation $y = 3x + 2$ form a straight line when plotted.

Explain to students that in the next activity they will be using their skills of substitution and solving equations to help them to investigate graphical representations of linear equations.

Collaborative activity: *Card Set A: Equations, Tables & Graphs* (20 minutes)

Organize students into pairs.

For each pair provide a cut up copy of *Card Set A: Equations, Tables & Graphs* and some plain paper.

These six cards each include a linear equation, a table of values and a graph. However, some of the information is missing.

In your pairs, share the cards between you and spend a few minutes, individually, completing them. You may need to do some calculations to complete the cards. Do these on the plain paper and be prepared to explain your method to your partner.

Once you have had a go at filling in the cards on your own, take turns to explain your work to your partner. Your partner should check your cards and challenge you if they disagree. It is important that you both understand and agree on the answers for each card.

When completing the graphs, take care to plot points carefully and make sure that the graph fills the grid in the same way as it does on Cards C1 and C3.

Slide P-1: *Card set A: Equations, Tables, Graphs* on the projector resource summarizes these instructions.

If students are struggling, suggest that they focus on Cards C4 and C5 first.

It does not matter if students are unable to complete all six cards. It is more important that they can confidently explain their strategies and have a thorough understanding of the skills they are using.

For students who complete all the cards successfully and need an extension, ask them to spend a few minutes comparing their completed cards:

Select two cards and note on your whiteboards any common properties of the equations and/or the graphs. Repeat this for all of your completed cards. This will help you later.

While students work in small groups you have two tasks: to make a note of students approaches to the task, and to support student reasoning.

Note student approaches to the task

Listen and watch students carefully and note any common mistakes. For example, are students misinterpreting the slope and intercept on cards where the graph has already been drawn? Do they fail to recognize an equation/graph that has a negative gradient? You may want to use the questions in the *Common issues* table to help address misconceptions.

Also notice the way in which students complete the cards. Do students use the completed table of values to plot the graph or do they use their knowledge of slope and intercept to draw the graph directly from the equation? Do students first plot the line using easy values for x or y, and then read off values from the graph in order to complete the table? Do students rearrange the equation or do they use guess and check to solve for x or y? Do students use multiplication to eliminate the fraction from the equation? Do students use the slope and intercept or guess and check to figure out the equation of the graph?

You will be able to use this information in the whole-class discussion.

Support student reasoning

Try not to make suggestions that move students towards a particular strategy. Instead, ask questions to help students to reason together.

Martha completed this card. Jordan, can you explain Martha's work?

Show me a different method from your partner's to check their method is correct.

If you find the student is unable to answer this question, ask them to discuss the work further. Explain that you will return in a few minutes to ask a similar question.

How can you check the card is correct? [Read off coordinates from the line, use the slope and intercept to check the equation matches the line, etc.]

For each card, encourage students to explain their reasoning and methods carefully.

How do you know that $y = 3$ when $x = 2$ in Card C2? What method did you use?

How did you find the missing equation on Card C1/C3? Show me a different method.

Suppose you multiply out the equation on Card C4. What information can you then deduce about the graph? [The y-intercept and slope.]

Which of these equations is arranged in a way that makes it easy to draw a graph using information about the line's y-intercept and slope? [C5.] What are they? [4 and -½.]

You may find some students struggle when the slope of a line is negative or when dealing with negative signs, or when the slope is a fraction.

Checking work (15 minutes)

Ask students to exchange their completed cards with another pair of students.

Carefully check the cards and point out any answers you think are incorrect.

You must give a reason why you think the card is incorrect but do not make changes to the card.

Once students have checked another group's cards, they need to review their own cards taking into account comments from their peers and make any necessary changes.

Extending the lesson over two days

If you are taking two days to complete the unit then you may want to end the first lesson here. At the start of the second day, briefly remind students of their previous work before moving on to the next collaborative activity.

Collaborative activity: Using *Card Set B* to link *Card Set A* (20 minutes)

Give each pair two copies of *Card Set B: Arrows* (already cut-up), a copy of *Graph Transparency*, a transparency pen, a large sheet of paper for making a poster, and a glue stick.

Choose two of your completed cards from Card Set A *and stick them on your poster paper with a gap in between.*

You are going to try and link these cards with one of the arrows.

The cards will either have no common solutions, one common solution or infinitely many common solutions. Select the appropriate arrow and stick it on your poster between the two cards. If the cards have one common solution, you will need to complete the arrow with the values of x and y where this solution occurs.

Add another completed card to your poster and compare it with the two already stuck down. Find arrows that link this third card with the other two and stick the cards down.

Continue to compare all the cards in this way, making as many links as possible. If some of the cards are incomplete, you will need to complete them before comparing them.

Slide P-2*: Card set B: Arrows* on the projector resource summarizes these instructions.

Some students might find it helpful to use a transparency when comparing the cards.

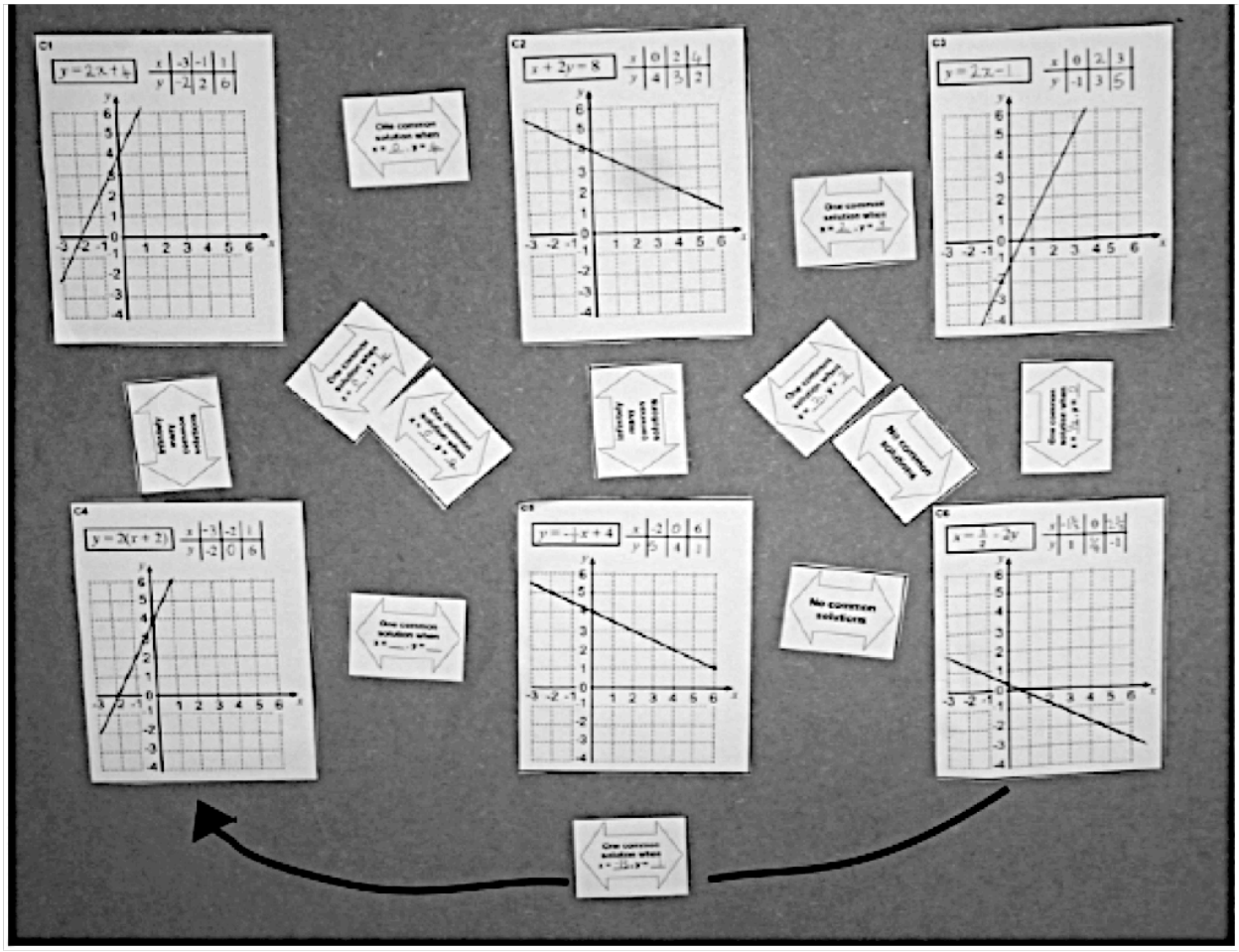
How might you use the Graph Transparency on your desk to help you to determine how many solutions the two cards have in common?

If students are struggling to identify how to use the transparency, ask them if tracing one of the graphs onto the transparency might be helpful. Some students may prefer to not use the transparency.

Notice how students are working and their method for completing the task. Are any students relying purely on the algebraic representation of the equation? Once students have recognized that there is one common solution, are they checking the solution algebraically as well as using the graphs?

As students work on the comparisons, support them as before. Again you may want to use some of the questions in the *Common issues*. Walk around and ask students to explain their decisions.

The finished poster produced may look like this:



Whole-class discussion (15 minutes)

Once groups have completed their posters, display them at the front of the room. Based on what you have learned about your students' strategies and the review of their posters, select one or two groups to explain how they went about addressing the task (if possible select groups who have taken very different approaches to the task). As groups explain their strategies, ask if anyone has a question for the group or if anyone used a similar strategy.

When a few groups have had a chance to share their approach, consolidate what has been learned. Using mini-whiteboards to encourage all students to participate, ask the following questions in turn:

1. *Show me two equations that have one common solution.*
   [E.g. $y = 2x + 4$ and $y = -\frac{1}{2}x + 4$].
   What are the solution values for x and y? [$x = 0$ and $y = 4$.]
   What happens to the graphs at this point? [They intersect each other.]
   On your mini-whiteboards make up two more equations that have one common solution. Don't use equations that appear on the cards. Sketch their graphs. Now show me!

2. *Show me two equations that have no common solutions.*
   [E.g. $y = 2x + 4$ and $y = 2x - 1$.]

How do you know they have no common solution?
[They are parallel lines so will never intersect.]

What do you notice about these two equations?
[They have the same coefficient of x/same slope.]
On your mini-whiteboards make up two more equations that have no common solutions. Don't use equations that appear on the cards. Sketch their graphs. Now show me!

3. *Show me two equations with infinitely many common solutions.*
   [E.g. $y = 2x + 4$ and $y = 2(x + 2)$.]
   What do you notice about the two graphs for these equations? [They are the same line.]
   Why is this? [$2(x + 2)$ is $2x + 4$ in factorized form.]

The focus of this discussion is to explore the link between the graphical representations of the equations and their common solutions, even though students may have used both the algebraic representation and the table of values during the classification process. Help students to recognize that solutions to a system of two linear equations in two variables correspond to the points of intersection of their graphs, as well as what it means graphically when there are no or infinitely many common solutions.

Follow-up lesson: *Working with Linear Equations (revisited)* (15 minutes)

Give back the responses to the original assessment task to students and a copy of the task *Working with Linear Equations (revisited.)*

Ask students to look again at their solutions to the original task together with your comments. If you have not added questions to individual pieces of work then write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

Look at your solutions to the original task Working with Linear Equations and read through the questions I have written.

Use what you have learned to answer these questions.

Using what you have learned, have a go at the second sheet: Working with Linear Equations (revisited).

Some teachers give this as a homework task.

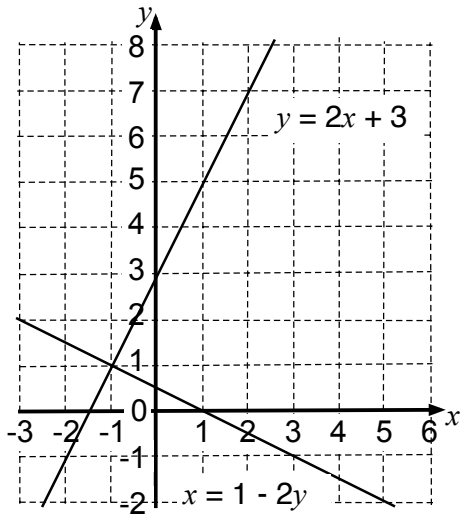
SOLUTIONS

Assessment task: *Working with Linear Equations*

1a. Tables A and D satisfy the equation $y = 2x + 3$.

Table B satisfies the equation $y = x + 5$ and table C is non-linear.

b.



$y = 2x + 3$

x	–2	0	1
y	–1	3	5

$x = 1 – 2y$

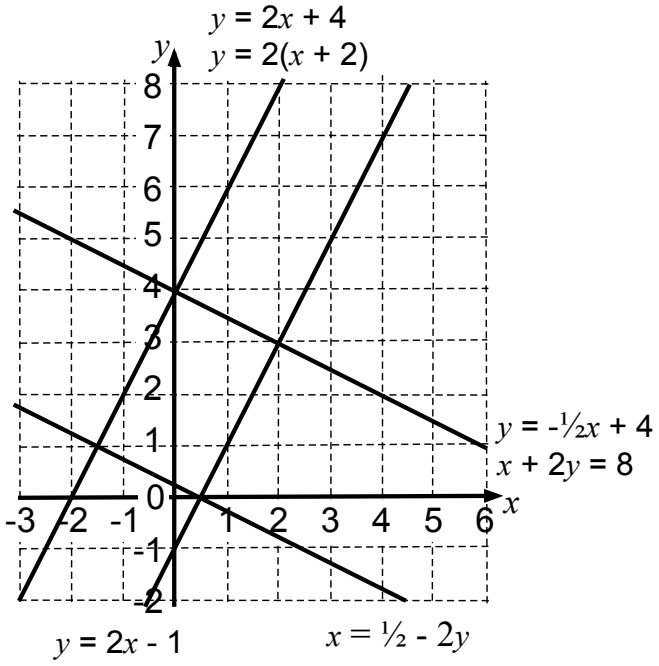
x	0	1	5
y	0.5	0	–2

c. The two graphs have one common solution at $x = –1$, $y = 1$. This is the point of intersection of the two graphs.

2. Students can draw any line that has the same slope as $y = 2x + 3$. For example $y = 2x$ or $y = 2x + 1$ etc.

Lesson task: *Card Sets A* and *B*

The six cards in *Card Set A* describe the four straight lines below:



C1

$y = 2x + 4$

x	–3	–1	1
y	–2	2	6

C2

$x + 2y = 8$

x	0	2	4
y	4	3	2

C3

$y = 2x – 1$

x	0	2	3
y	–1	3	5

C4

$y = 2(x + 2)$

x	–3	–2	1
y	–2	0	6

C5

$y = - ½x + 4$

x	–2	0	6
y	5	4	1

C6

$x = ½ – 2y$

x	–1.5	0	2.5
y	1	0.25	–1

Infinitely many common solutions

$x + 2y = 8$ and $y = – ½x + 4$	$x + 2y = 8$ (C2) is a rearrangement of $y = –½x + 4$.
$y = 2(x + 2)$ and $y = 2x + 4$	$y = 2(x + 2)$ (C4) is the factorized form of $y = 2x + 4$.

No common solutions

$y = 2x + 4$ and $y = 2x – 1$	Equal slopes.
$y = –½x + 4$ and $x = ½ – 2y$	Equal slopes.

One common solution

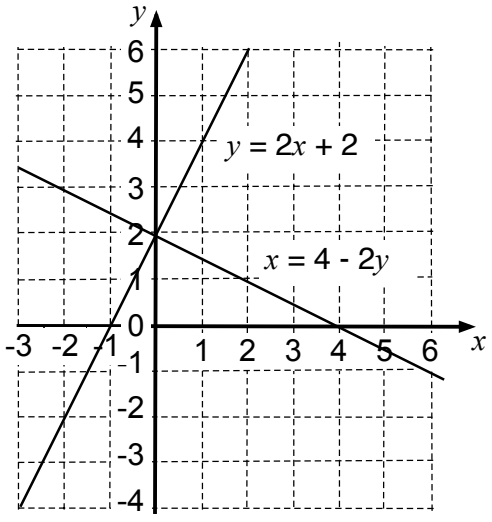
$y = 2x + 4$ (or $y = 2(x + 2)$) and $y = –½x + 4$ (or $x + 2y = 8$)	have one common solution at (0,4).
$y = 2x + 4$ (or $y = 2(x + 2)$) and $x = ½ – 2y$	have one common solution at (–1.5,1).
$y = 2x – 1$ and $y = –½x + 4$ (or $x + 2y = 8$)	have one common solution at (2,3).
$y = 2x – 1$ and $x = ½ – 2y$	have one common solution at (0.5,0).

Assessment task: *Working with Linear Equations (revisited)*

1a. Tables B and D satisfy the equation $y = 2x + 2$.

Table A is non-linear and table C satisfies the equation $y = 3x + 1$.

b.



$y = 2x + 2$

x	–3	0	2
y	–4	2	6

$x = 4 – 2y$

x	2	4	6
y	1	0	–1

c. The two graphs have one common solution at $x = 0$, $y = 2$. This is the point of intersection of the two graphs.

2. Students can draw any line that has the same slope as $y = 2x + 2$. For example $y = 2x$ or $y = 2x + 1$ etc.

Working with Linear Equations

x	-3	2	3
y	-3	7	9

A

x	0	2	4
y	5	7	9

B

x	-1	0	2
y	5	1	7

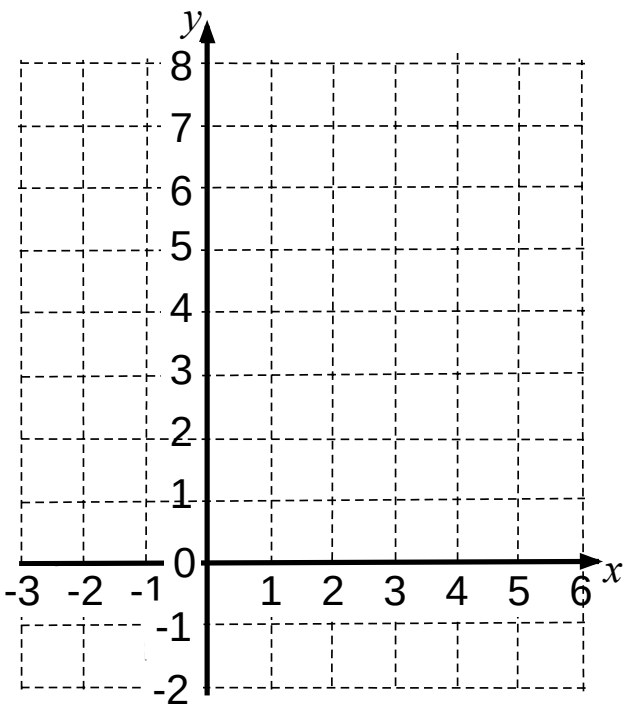
C

x	-1	0	2
y	1	3	7

D

1a. Which of these tables of values satisfy the equation $y = 2x + 3$? Explain how you checked.

b. By completing the table of values, draw the lines $y = 2x + 3$ and $x = 1 – 2y$ on the grid.



$y = 2x + 3$

x	-2	0	
y			5

$x = 1 – 2y$

x	0		5
y		0	

c. Do the equations $y = 2x + 3$ and $x = 1 – 2y$ have one common solution, no common solutions, or infinitely many common solutions? Explain how you know.

2. Draw a straight line on the grid that has no common solutions with the line $y = 2x + 3$. What is the equation of your new line? Explain your answer.

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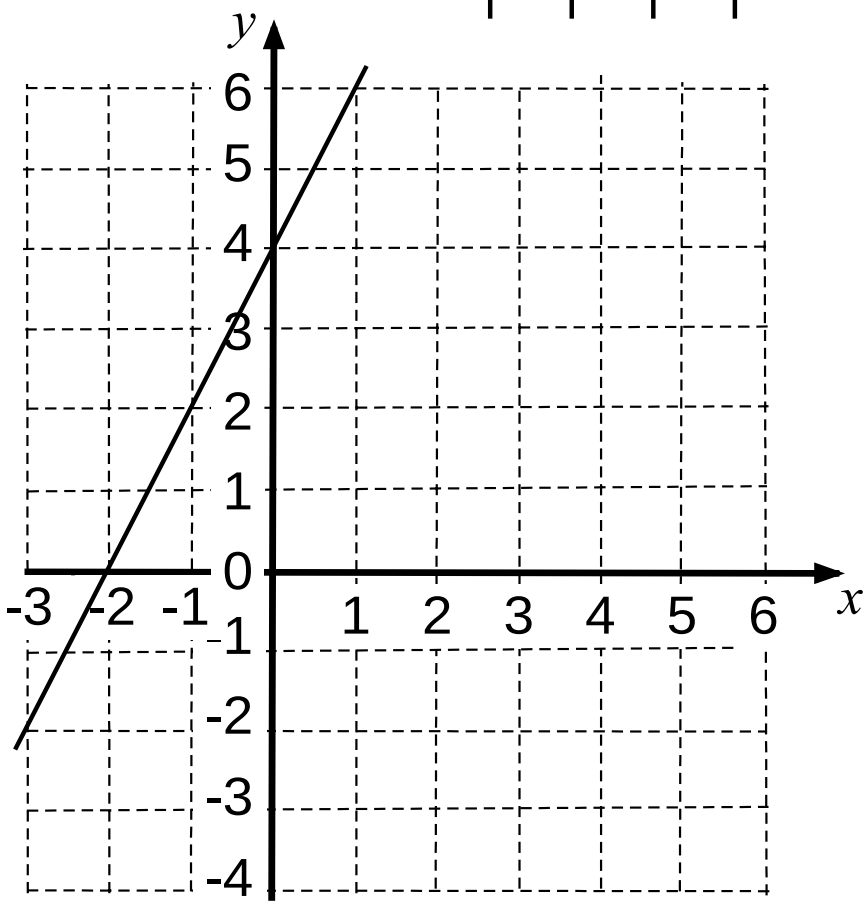
Card Set A: Equations, Tables & Graphs

C1

$y =$ _ _ _ _ _ _

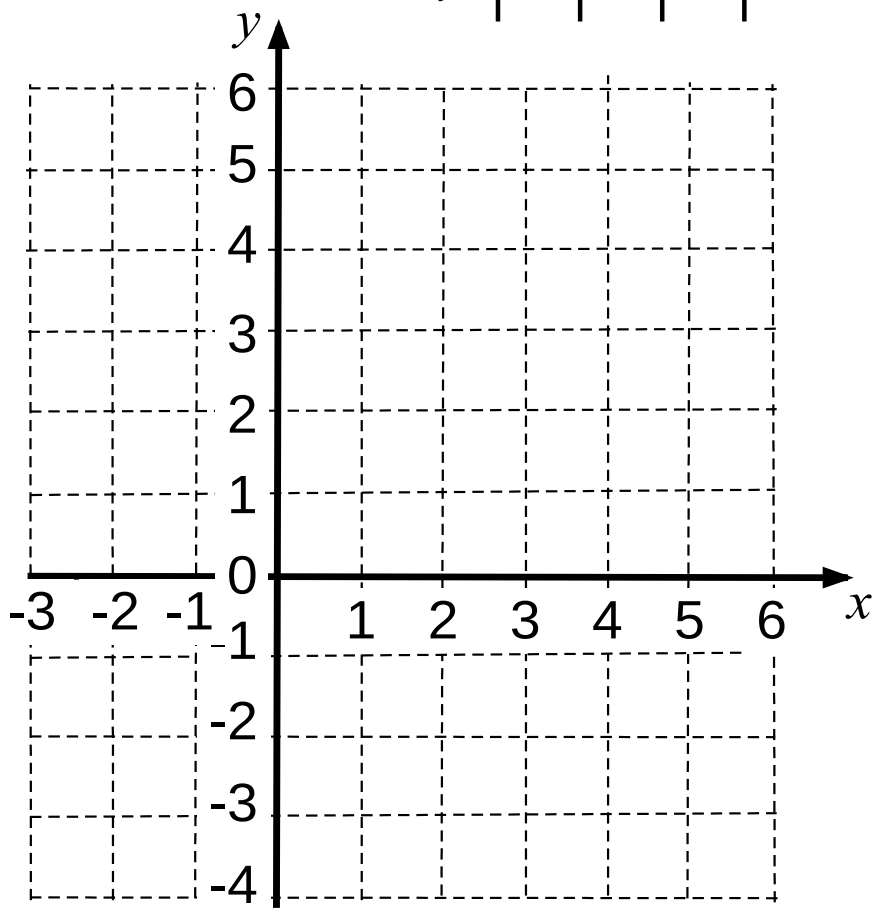
x	-3		1
y		2	



C2

$x + 2y = 8$

x	0	2	
y	4		2



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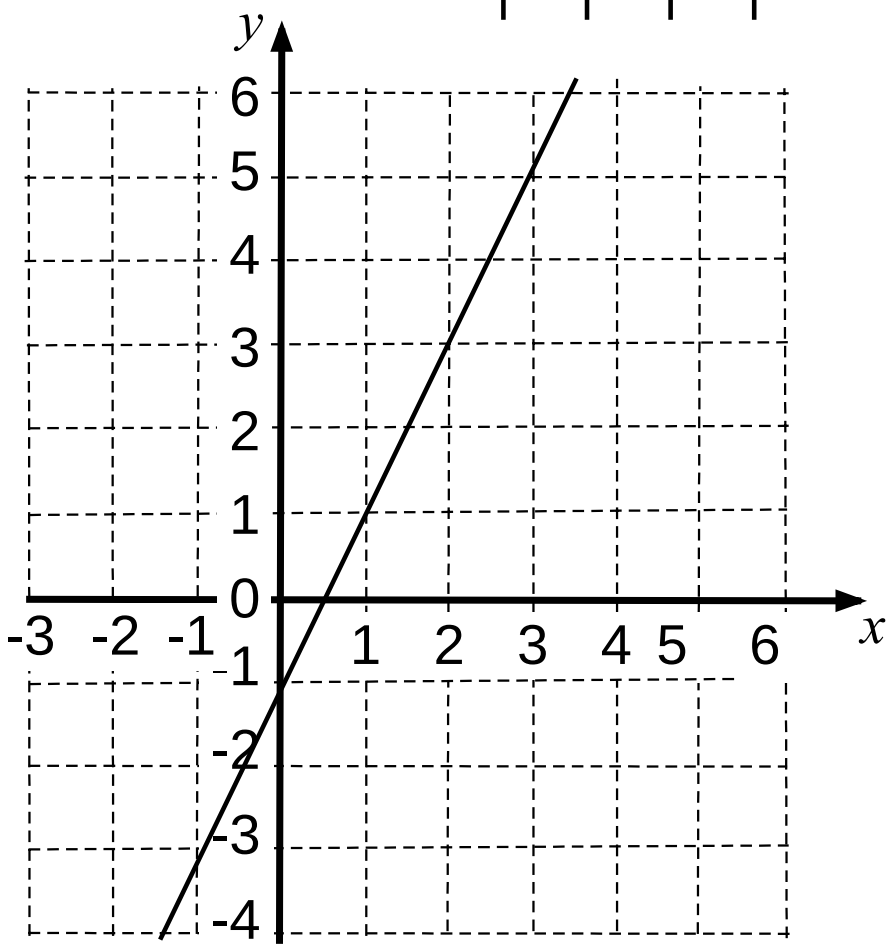
Card Set A: Equations, Tables & Graphs (continued)

C3

$y =$ _ _ _ _ _ _ _

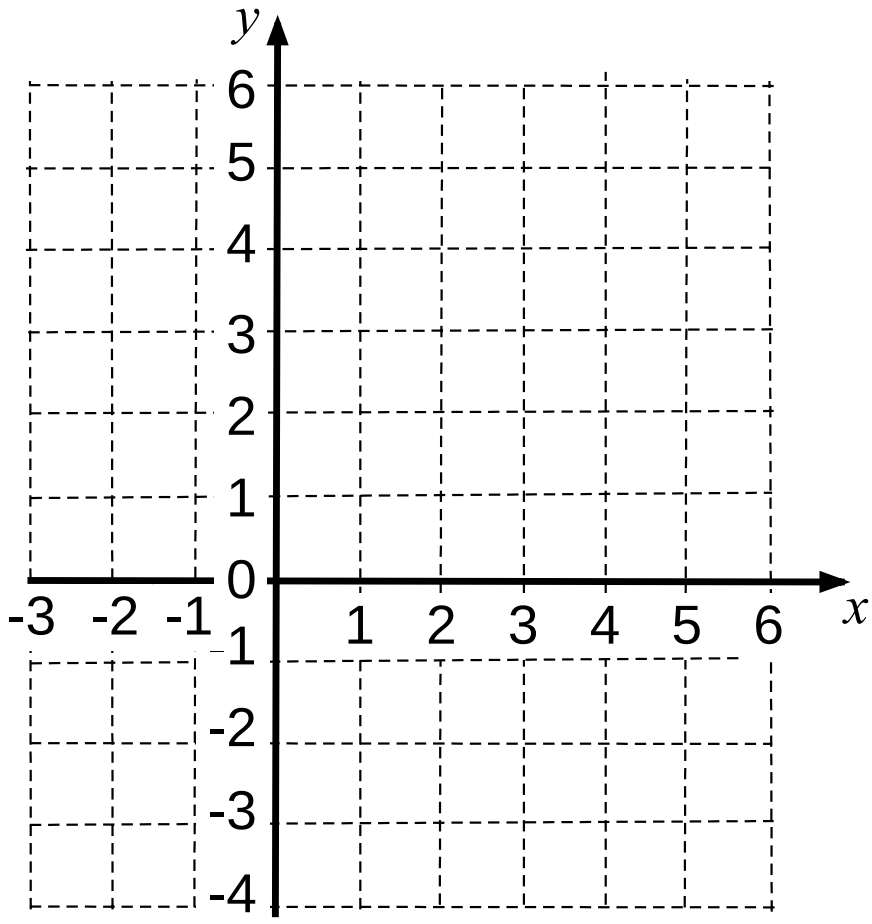
x	0		3
y	-1	3	



C4

$y = 2(x + 2)$

x		-2	
y	-2		6



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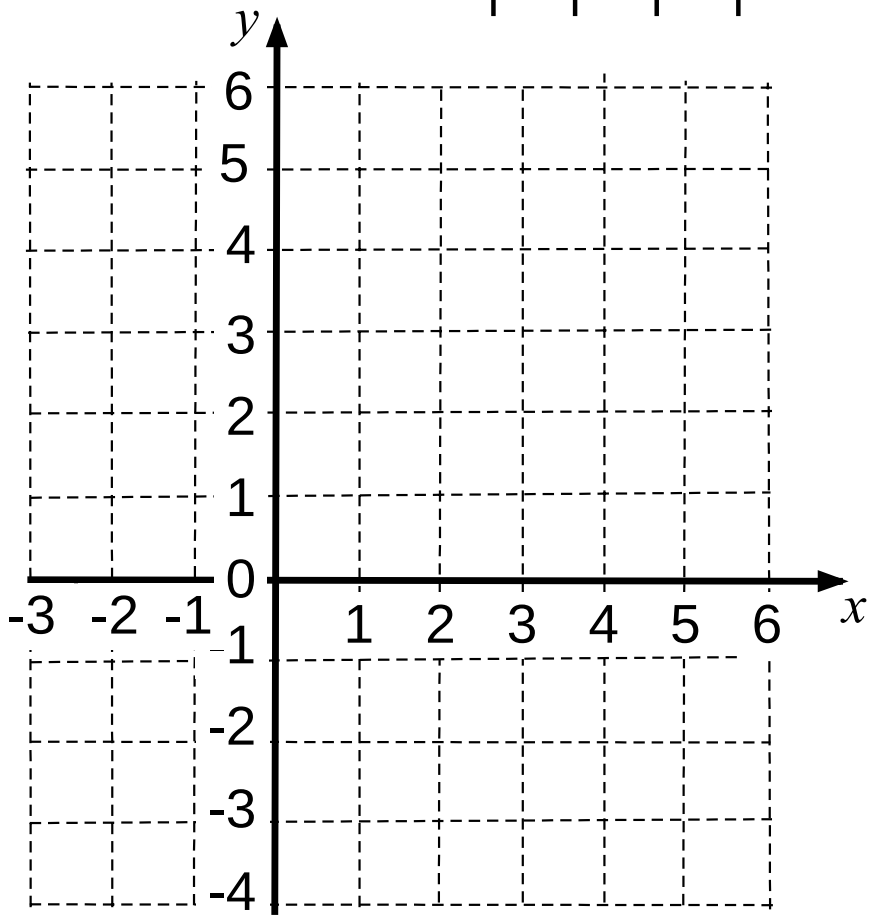
Card Set A: Equations, Tables & Graphs (continued 2)

C5

$y = -\frac{1}{2}x + 4$

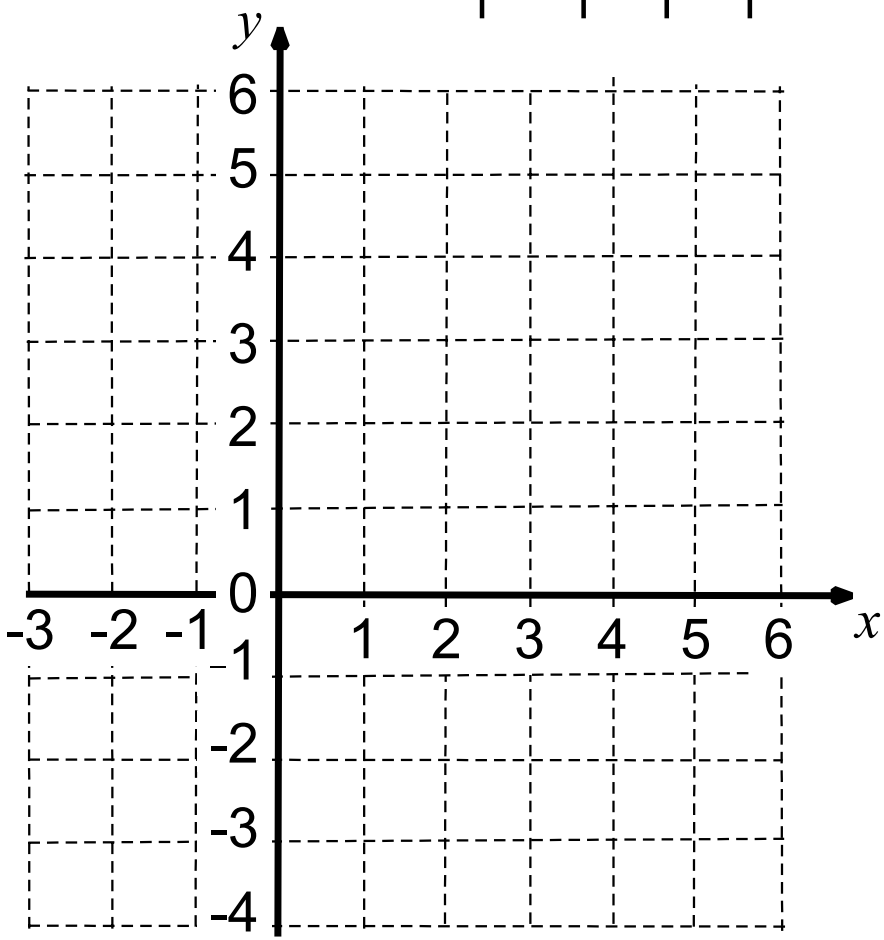
x	-2		6
y		4	



C6

$x = \frac{1}{2} - 2y$

x		0	
y	1		-1



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Card Set B: Arrows

No common solutions	No common solutions	Infinitely many common solutions	Infinitely many common solutions
One common solution when $x =$ ___ , $y =$ ___	One common solution when $x =$ ___ , $y =$ ___	One common solution when $x =$ ___ , $y =$ ___	One common solution when $x =$ ___ , $y =$ ___
One common solution when $x =$ ___ , $y =$ ___	One common solution when $x =$ ___ , $y =$ ___	One common solution when $x =$ ___ , $y =$ ___	One common solution when $x =$ ___ , $y =$ ___

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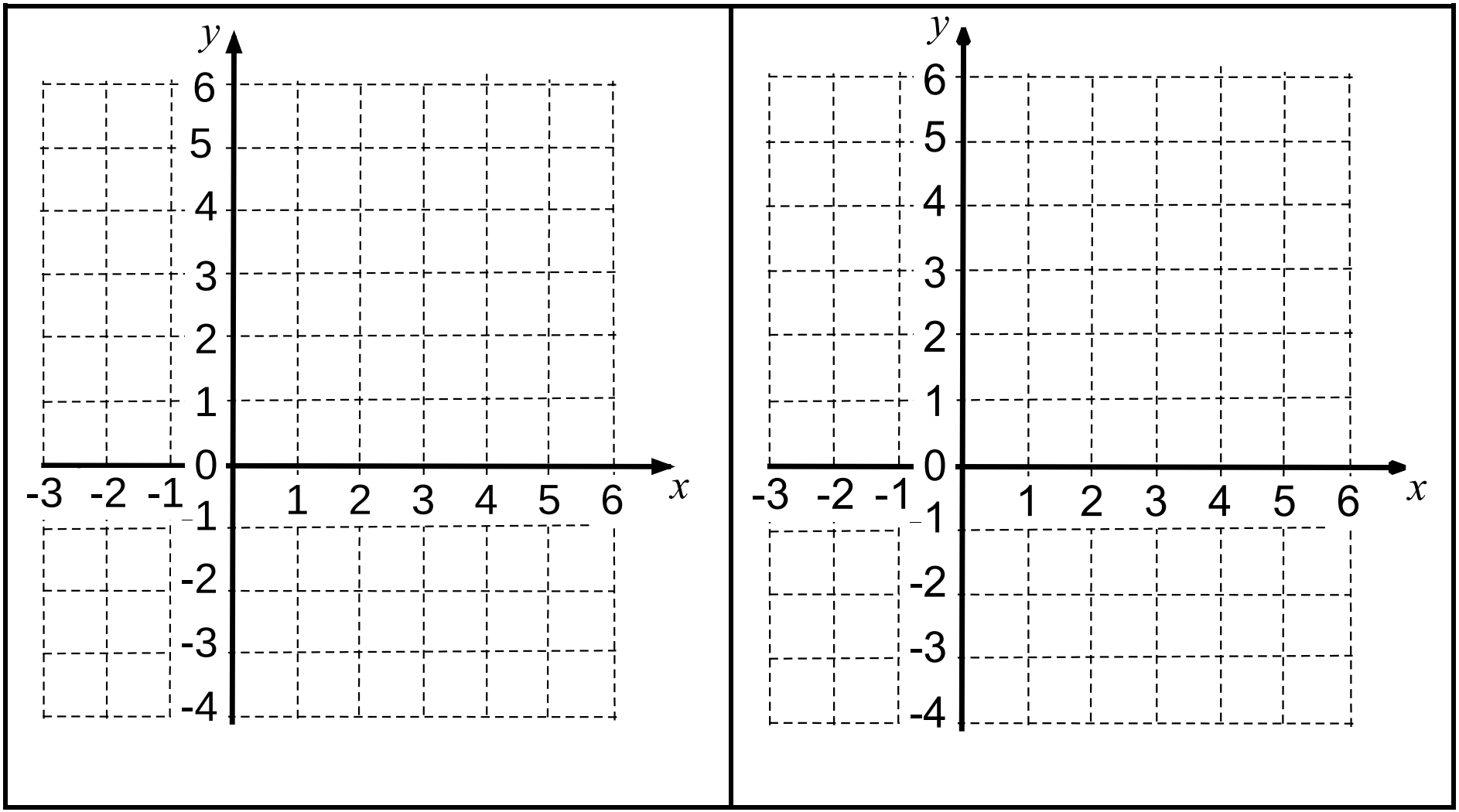
Graph Transparency



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Working with Linear Equations (revisited)

x	0	1	2
y	4	4	5

A

x	1	2	3
y	4	6	8

B

x	-2	-1	1
y	-5	-2	4

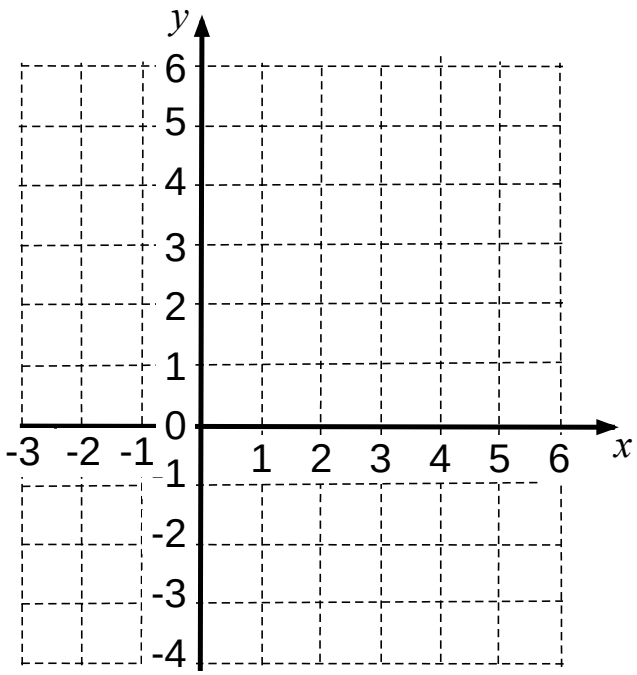
C

x	-2	-1	1
y	-2	0	4

D

1a. Which of these tables of values satisfy the equation $y = 2x + 2$? Explain how you checked.

b. By completing the table of values, draw the lines $y = 2x + 2$ and $x = 4 – 2y$ on the grid.



$y = 2x + 2$

x	-3	0	
y			6

$x = 4 – 2y$

x		4	6
y	1		

c. Do the equations y = 2x + 2 and x = 4 – 2y have one common solution, no common solutions, or infinitely many common solutions? Explain how you know.

2. Draw a straight line on the grid that has no common solutions with the line $y = 2x + 2$.
What is the equation of your new line? Explain your answer.

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Card Set A: Equations, Tables, Graphs

1. Share the cards between you and spend a few minutes, individually, completing the cards so that each has an equation, a completed table of values and a graph.

2. Record on paper any calculations you do when completing the cards. Remember that you will need to explain your method to your partner.

3. Once you have had a go at filling in the cards on your own:
   - Explain your work to your partner.
   - Ask your partner to check each card.
   - Make sure you both understand and agree on the answers.

4. When completing the graphs:
   - Take care to plot points carefully.
   - Make sure that the graph fills the grid in the same way as it does on Cards C1 and C3.

Make sure you both understand and agree on the answers for every card.

Card Set B: Arrows

1. You are going to link your completed cards from *Card Set A* with an arrow card.

2. Choose two of your completed cards and decide whether they have no common solutions, one common solution or infinitely many common solutions. Select the appropriate arrow and stick it on your poster between the two cards.

3. If the cards have one common solution, complete the arrow with the values of x and y where this solution occurs.

4. Now compare a third card and choose arrows that link it to the first two. Continue to add more cards in this way, making as many links between the cards as possible.

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
Shell Center Team at the Center for Research in Mathematical Education
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with
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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org

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