PROBLEM SOLVING

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Generalizing Patterns: *The Difference of Two Squares*

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Generalizing Patterns: *The Difference of Two Squares*

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students working with square numbers are able to:

- Choose an appropriate, systematic way to collect and organize data, examining the data for patterns.
- Describe and explain findings clearly and effectively.
- Generalize using numerical, geometrical, graphical and/or algebraic structure.
- Explain why certain results are possible/impossible, moving towards a proof.

COMMON CORE STATE STANDARDS

This lesson relates to **all** *Mathematical Practices* in the *Common Core State Standards for Mathematics*, with a particular emphasis on:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

8.EE: Work with radicals and integer exponents.
8.F: Define, evaluate, and compare functions.
Use functions to model relationships between quantities.

INTRODUCTION

This lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task designed to reveal their current understanding and difficulties. You review their responses and create questions for them to consider when improving their work.
- At the start of the lesson, students reflect on their individual responses and use the questions posed to think of ways to improve their work. They then work collaboratively in small groups to produce, in the form of a poster, a better solution to the task than they did individually.
- In a whole-class discussion students compare and evaluate the different strategies they have used.
- Working in the same small groups, students analyze sample responses to the task.
- In a whole-class discussion, students review the methods they have seen.
- At the end of the lesson, or in a follow-up lesson, students reflect individually on their work.

MATERIALS REQUIRED

- Each student will need a copy of the assessment task: *The Difference of Two Squares*, some plain paper, and a copy of the *How Did You Work?* review questionnaire.
- Each small group of students will need a large sheet of paper for making a poster, some felt-tipped pens and copies of the *Sample Responses to Discuss*.
- There is a projector resource to support whole-class discussions.

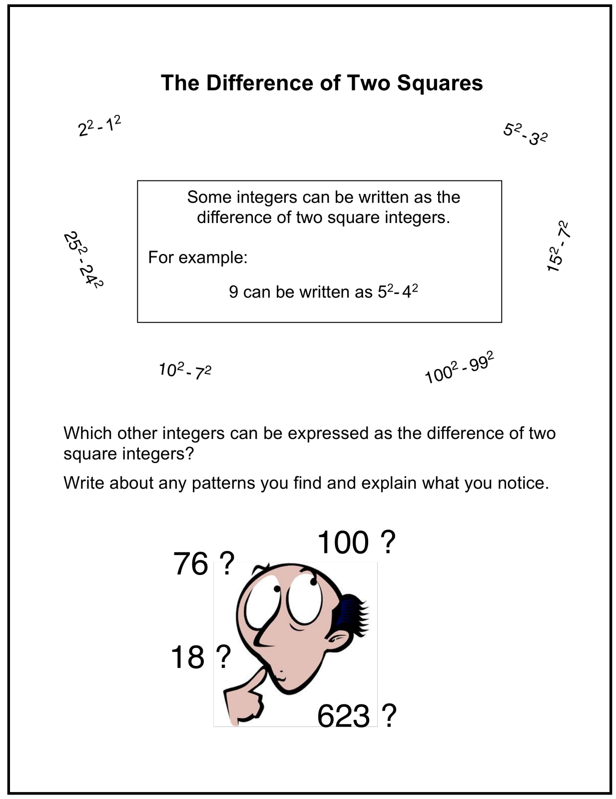
TIME NEEDED

20 minutes before the lesson, an 80-minute lesson (or split into two shorter lessons), and 15 minutes in a follow-up lesson (or for homework). Timings are approximate and will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *The Difference of Two Squares* (20 minutes)

Have students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work, and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.



The Difference of Two Squares

$2^2 - 1^2$

$5^2 - 3^2$

$25^2 - 24^2$

Some integers can be written as the difference of two square integers.

For example:

9 can be written as $5^2 - 4^2$

$15^2 - 7^2$

$10^2 - 7^2$

$100^2 - 99^2$

Which other integers can be expressed as the difference of two square integers?

Write about any patterns you find and explain what you notice.

Give each student a copy of the assessment task: *The Difference of Two Squares* and some plain paper to work on.

You may first want to check students' understanding of the term 'integer'.

> *Read the task carefully and then use the plain paper to answer the questions.*

It is important that, as far as possible, students are allowed to answer the questions without assistance. Some students may find it difficult to get started: be aware that if you offer help too quickly, students are likely to do what you say and will not think for themselves. If, after several minutes, students are still struggling, try to help them understand what is required.

Students who sit together often produce similar answers, and then when they come to compare their work, they have little to discuss. For this reason, we suggest that when students do the task individually, you ask them to move to different seats. Then at the beginning of the formative assessment lesson, allow them to return to their usual seats. Experience has shown that this produces more profitable discussions.

When all students have made a reasonable attempt at the task, reassure them that they will have time to revisit and revise their solutions later.

Assessing students' responses

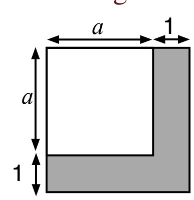
Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem solving approaches. We suggest that you do not score students' work. Research shows that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- Write one or two questions on each student's work, or
- Give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students, and write these on the board when you return the work to the students.

Common issues	Suggested questions and prompts
Considers only the examples provided in the task or considers a limited number of examples	• Think of other pairs of square numbers you could find. • What do you notice about the calculations in the examples? Investigate other different, but similar calculations.
Uses an unsystematic approach For example: The student finds the differences for a variety of square integers but, due to a lack of structure, is unable to make any conclusions about the integers generated.	• Can you group calculations that have something in common? • Can you vary the numbers systematically? • Can you organize your results in some way to help to identify where there are gaps? • Can you find a pattern from the examples you have written down?
Provides little or no explanation For example: The student notices that all odd numbers can be expressed as the difference of two consecutive square integers but does not provide an adequate explanation.	• If the integers being squared are consecutive, what properties will the two integers have? • Can you explain **why** the difference of two consecutive square integers is an odd integer? • Can you write an expression, in terms of a, for the grey area in this diagram?  • Can you use this diagram to help explain why only odd integers are generated from the difference of two consecutive square integers? • Use algebra to explain your work.

Common issues	Suggested questions and prompts
Makes inaccurate assumptions For example: The student notices that adding consecutive numbers gives the same result as squaring and subtracting and assumes that this works for all pairs of integers, and so looks at finding integers that can be generated by adding rather than squaring and subtracting. E.g. $4 + 5 = 5^2 - 4^2$ (correct) $4 + 2 = 4^2 - 2^2$ (incorrect).	• What assumption have you made? • What does $3^2 - 2^2$ equal? What does $3 + 2$ equal? What do you notice? Does this always work? Why? • Does, (for example) $5^2 - 3^2 = 5 + 3$? Why/why not? • Can you find a similar rule that works for integers with a difference of 2, 3 or 4? • Can you use what you have found to help to identify which integers can be expressed as the difference of two square integers?
Provides little or no justification For example: The student identifies that some integers cannot be expressed as the difference of two squares having tried a few combinations.	• How could you convince me that no combination of square integers will produce the integer(s) 2, 6, 10, 14…? • Use algebra to help explain your work.
Considers positive integers only For example: The student concludes that integers of the form $4n - 2$ cannot be expressed as the difference of two square integers but that all other positive integers can.	• Does what you have found for positive integers extend to negative integers? • What rules apply when squaring negative numbers?

SUGGESTED LESSON OUTLINE

Review individual solutions to the problem (10 minutes)

Return the assessment task to the students. If you did not add questions to individual pieces of work, write your list of questions on the board. Students are to select questions appropriate to their own work, and spend a few minutes thinking about them.

Begin the lesson by briefly reintroducing the problem. Display Slide P-1 of the projector resource.

Recall the Difference of Two Squares problem. What was the task about?

Today you are going to work together to try to improve your initial attempts at this task. I have had a look at your work and have some questions I would like you to think about.

On your own, carefully read through the questions I have written. I would like you to use the questions to help you to think about ways of improving your own work.

Make a note of anything you think will help to improve your work.

Collaborative small-group work (30 minutes)

Organize the class into small groups of two or three students. Give each group a large sheet of paper, and some felt-tipped pens.

Deciding on a Strategy

Ask students to share their ideas about the task and plan a joint solution.

You each have your own individual solution and have been thinking about how you might improve this. I want you to share your work with your group and your ideas for improving it.

Listen carefully to each other and ask questions if you don't understand or agree.

Once students have evaluated the relative merits of each approach, ask them to agree on a strategy and write an outline of what they plan to do on their large sheet of paper.

Once you understand each other's work, agree together in your group on the best approach for completing the problem.

Outline on your large sheet of paper the approach you are going to use.

Slide P-2 of the projector resource, *Planning a Joint Method*, summarizes these instructions.

Planning a Joint Method

1. Take turns to explain your work and how you think it could be improved.
2. Listen carefully to each other and ask questions if you don't understand or agree.
3. Once everyone in the group has explained their method, plan a joint method that is better than each of your separate ideas.
4. Make sure that everyone in the group can explain the reasons for your chosen method.
5. Write a brief outline of your planned method on your large sheet of paper.

Implementing the Strategy

Turn over your large sheet of paper and on the other side produce a poster that shows a joint solution to the task. State on your poster clear reasons for your choice of method.

While students work in small groups you have two tasks: to note different student approaches to the task and to support student problem solving.

Note different student approaches to the task

Listen and watch students carefully. Note different approaches to the task and whether or not they make any assumptions. Do students work systematically? How do students explain why certain integers are impossible to express as the difference of two squares? Do students attempt to generalize? Do students make assumptions? Do students use algebra?

Are students checking their work? If they are, what checking methods are they using? In particular, note any common mistakes. You can then use this information to focus a whole-class discussion towards the end of the lesson.

Support student problem solving

Try not to make suggestions that move students towards a particular approach to the task. Instead, ask questions that help students to clarify their thinking. In particular focus on the strategies rather than the solution. Encourage students to justify their solutions.

What is your strategy? What do you need to do next?

What have you found out so far? Can you explain why this is?

How can you systematically consider the possibilities?

How can you check your work?

Have you made any assumptions?

You may want to use the questions in the *Common issues* table to support your questioning. If the whole class is struggling on the same issue, you could write one or two relevant questions on the board or hold a brief whole-class discussion.

Sharing different approaches (10 minutes)

Hold a whole-class discussion on the strategies used to produce a group solution. Ask two or three groups of students with contrasting approaches to present their posters and describe the method used. As part of their presentation, ask students to describe the ways in which their strategy differs from their initial individual responses.

What method did you use? In what ways was this strategy different to your individual approach?

How did looking at other strategies in your group influence your thinking?

It is not necessary at this stage to focus attention on what their strategy showed, but instead to help students to make sense of different approaches by listening to and questioning their peers. This will help students with the next activity where they will be evaluating two different approaches to the task.

Extending the lesson over two days

If you are taking two days to complete the unit then you may want to end the first lesson here. At the start of the second day, briefly remind students of their previous work before moving on to the collaborative analysis of sample responses.

Collaborative analysis of *Sample Responses to Discuss* (15 minutes)

Distribute copies of the *Sample Responses to Discuss* to each group of students.

This task gives students an opportunity to evaluate different ways of approaching the task, without providing a complete solution strategy.

In your groups you are now going to look at two pieces of student work on the task. Notice in which ways this work is similar to yours and in which ways it is different.

There are some questions for you to answer as you look at the work. You may want to add notes to the work to make it easier to follow.

Slide P-3 of the projector resource, Sample Responses to Discuss, describes how students should work together.

> **Sample Responses to Discuss**
>
> 1. Each choose a sample response to work through and write your answers on your mini-whiteboards.
> 2. Explain your answers to the rest of the group.
> 3. Listen carefully to explanations. Ask questions if you don't understand.
> 4. Once everyone is satisfied with the explanations, write the answers to the questions below the sample response.
> 5. Try to ensure that the student who writes the answers is not the student who explained them.

During the small group work, support the students as in the first collaborative activity. Also, check to see which method students find more difficult to understand. Note similarities and differences between the sample approaches and those the students took in the collaborative group work.

Aakar uses a two-way table. He correctly notices patterns along some of the diagonals. There are, however, some patterns he has missed: (e.g. the odd numbers of the form $6x$-9 and the multiples of 8). Aakar has not noticed anything special about the numbers within the expressions that make the diagonal numbers. He has also not noticed the number patterns in the rows.

Aakar has not considered subtractions that result in negative integers being produced. He has left this part of the two-way table blank.

Aakar has made no attempt to generalize.

Aakar could improve his work by using algebra to describe the patterns produced in the two-way table.

I will start by constructing a table so that I can check I have looked at the integers produced by all possible combinations.

Difference of two squares : $x^2 - y^2$

x^2-y^2 (y \ x)	0	1	2	3	4	5	6	
0	0	1	$\boxed{4}$	9	16	25	36	
1		0	3	$\boxed{8}$	15	24	35	
2			0	5	$\boxed{12}$	21	32	
3				0	7	$\boxed{16}$	27	
4					0	9	$\boxed{20}$	multiples of 4
5						0	11	odd numbers
6							0	

Replacing the y in the expression $x^2 - y^2$ with an expression in terms of x gives the following expressions for each of the diagonals:

Difference of 1: $x^2 - (x - 1)^2 = 2x - 1$.
Difference of 2: $x^2 - (x - 2)^2 = 4x - 4$.
Difference of 3: $x^2 - (x - 3)^2 = 6x - 9$.
Difference of 4: $x^2 - (x - 4)^2 = 8x - 16$ etc.

If the difference between the two numbers is n, then the expression for the integers produced can be written in the form $2nx - n^2$, assuming that the pattern continues.

For each row there is also a number pattern which can be written in terms of x to give x^2 for the first row, then $x^2 - 1$, $x^2 - 4$, $x^2 - 9$, $x^2 - 16$ etc.

Heng uses areas to explain his answer.

Heng has split each large square into rectangles. One of these rectangles always has an area of 1, another an area of $(n - 1)^2$, where n is the length of a side of the large square.

Area first square large square = 2^2.

$2^2 - 1^2 = 2 \times 1 + 1$

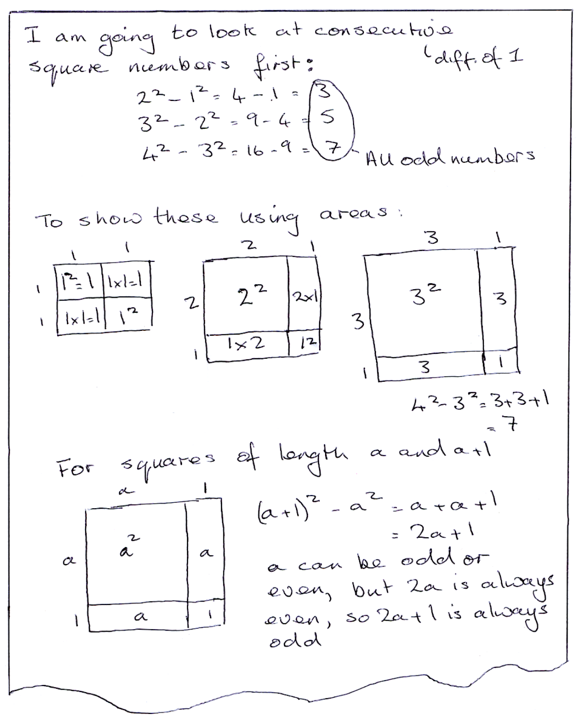
Area second large square = 3^2.

$3^2 - 2^2 = 2 \times 2 + 1$

Area third large square = 4^2.

$4^2 - 3^2 = 2 \times 3 + 1$

The +1 in each calculation means the difference of the consecutive square numbers is an odd number.



Heng has split the large square into smaller rectangles. What do they have in common?

Write a calculation for each of the areas of first two squares that will help explain Heng's thinking.

Heng correctly explains why the answer is always odd when the numbers in the expression are consecutive. He has not attempted to consider any other combination of numbers in the expression, such as when the integers differ by 2, 3 and so on.

Heng could add some shading to his area diagrams to make it clear the area being considered. He could continue his work to look at integers that are not consecutive. He could then extend his argument to any numbers, n and m say.

Whole-class discussion: comparing different approaches (15 minutes)

Hold a whole-class discussion to consider the different approaches used in the sample work. Focus the discussion on parts of the task students found difficult. Ask the students to compare the different solution methods.

Which approach did you like best? Why?

Which approach did you find most difficult to understand? Why?

Which approach provides more information? Why?

What did Aakar's/Heng's approach show? Did it allow him to identify which integers can be expressed as the difference of two squares?

To support the discussion, you may want to use Slides P-4 and P-5 of the projector resource.

Follow-up lesson (or possible homework): individual reflection (15 minutes)

Give out the sheet *How Did You Work?* and ask students to complete this questionnaire. The questionnaire should help students review their own progress.

Think carefully about your work in this lesson and the different methods you have seen and used. Spend a few minutes on your own answering the questions.

If you have time, ask your students to read through their original solution again and using what they have learned, have another go at completing the task. In this case, give each student a blank copy of the assessment *The Difference of Two Squares*.

SOLUTIONS

Assessment task: *The Difference of Two Squares*

There are many ways that students could approach this task. Below are a few possible examples. Proofs of conjectures will usually involve manipulating quadratic functions, which may be beyond many students at this level. However, systematic investigations that develop patterns and that show an exhaustive consideration of cases may be possible.

Students may choose to construct a table:

Difference of 1	Difference of 2	Difference of 3	Answer
$1^2 - 0^2$			1
Cannot be expressed as the difference of two squares.			2
$2^2 - 1^2$			3
	$2^2 - 0^2$		4
$3^2 - 2^2$			5
Cannot be expressed as the difference of two squares.			6
$4^2 - 3^2$			7
	$3^2 - 1^2$		8
$5^2 - 4^2$		$3^2 - 0^2$	9
Cannot be expressed as the difference of two squares.			10
$6^2 - 5^2$			11
	$4^2 - 2^2$		12
$7^2 - 6^2$			13
Cannot be expressed as the difference of two squares.			14
$8^2 - 7^2$		$4^2 - 1^2$	15
	$5^2 - 3^2$		16

From the table students may notice that all odd integers can be expressed as the difference of two consecutive square integers. Integers that are multiples of 4 can also be expressed as the difference of two squares. The two integers in these expressions differ by 2.

From the table, it may be conjectured that the only integers that cannot be expressed as the difference of two squares are those that are of the form $4n - 2$.

To explain why certain integers cannot be expressed as the difference of two squares, some students may try to argue systematically, for example:

For the integer 2, Try $2^2 - 0^2 = 4$, or $2^2 - 1^2 = 3$. Neither of these expressions makes 2.

Try $3^2 - 0^2 = 9$, or $3^2 - 1^2 = 8$, or $3^2 - 2^2 = 5$. None of these expressions make 2.

As the first number in the expression gets greater then the answer will also get greater. All answers will be greater than 2.

From the table you or/and your students may notice that the numbers that are twice an odd number cannot be expressed as the difference of two squares. You may be interested in the following argument that supports this:

Suppose we can write $n = a^2 - b^2$,
where n is twice an odd number and a and b are positive integers.

As n is twice an odd number, it is even.

This implies that a^2 and b^2 are either both even or both odd.

This implies that a and b are either both even or both odd.

This implies that $(a - b)$ and $(a + b)$ are both even.

But $n = (a - b)(a + b)$, so n must be divisible by 4.

But n is twice an odd number, which is not a multiple of 4. This is a contradiction.

Therefore twice an odd number cannot be a difference of two squares.

Students may follow the approach of Aakar, and use a two-way table to display $x^2 - y^2$:

x

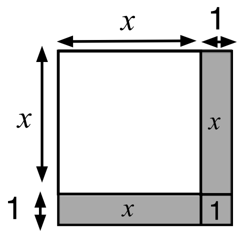
y	0	1	2	3	4	5	6	Diagonal patterns	Horizontal patterns
0	0	1	4	**9**	16	25	36		x^2
1		0	3	8	**15**	24	35	Multiples of 5, $x^2 - (x - 5)^2 = 10x - 25$	$x^2 - 1^2$
2			0	5	12	**21**	32	Multiples of 8, $x^2 - (x - 4)^2 = 8x - 16$	$x^2 - 2^2$
3				0	7	16	**27**	**Go up in 6's, $x^2 - (x - 3)^2 = 6x - 9$**	$x^2 - 3^2$
4					0	9	20	Multiples of 4, $x^2 - (x - 2)^2 = 4x - 4$	$x^2 - 4^2$
5						0	11	Odd numbers, $x^2 - (x - 1)^2 = 2x - 1$	$x^2 - 5^2$
6							0		

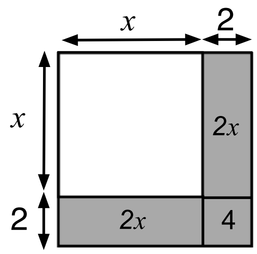
The table shows how some integers can be expressed as the difference of two squares. The algebraic expressions in the table all assume the patterns continue.

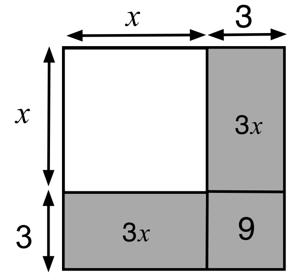
Diagonal patterns:

Students may notice a pattern in the structure of these expressions: if the difference between the two numbers is n, then the expression can be written in the form $2nx - n^2$. This argument is not a proof as it assumes the pattern continues.

Students may also use area diagrams:







Area = $(x + 1)^2 - x^2 = 2x + 1$

This shows all odd numbers can be expressed as the difference of two squares.

Area = $(x + 2)^2 - x^2 = 4x + 4$

This shows all multiples of 4 can be expressed as the difference of two squares.

Area = $(x + 3)^2 - x^2 = 6x + 9$

This shows all numbers of this form expressed as the difference of two squares.

Students may then assume that when the difference between the two numbers is n the integer answer is $2nx + n^2$. To help justify why certain integers cannot be expressed as the difference of two squares some students may use the fact that as x increases the gray areas will always increase.

The Difference of Two Squares

$2^2 - 1^2$

$5^2 - 3^2$

Some integers can be written as the difference of two square integers.

For example:

9 can be written as $5^2 - 4^2$

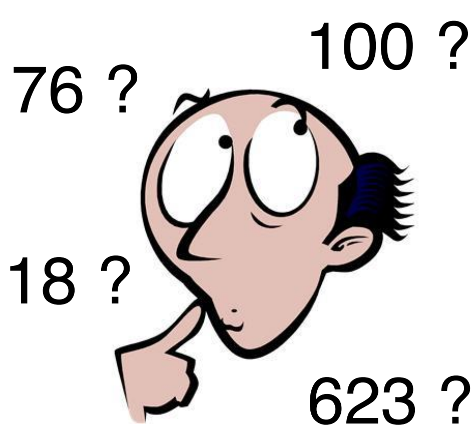
$25^2 - 24^2$

$15^2 - 7^2$

$10^2 - 7^2$

$100^2 - 99^2$

Which other integers can be expressed as the difference of two square integers?
Write about any patterns you find and explain what you notice.



Sample Responses to Discuss: Aakar

I will start by constructing a table so that I can check I have looked at the integers produced by all possible combinations.

Difference of two squares : $x^2 - y^2$

	x^2-y^2	x: 0	1	2	3	4	5	6	
y	0	0	1	$\boxed{4}$	9	16	25	36	
	1		0	3	$\boxed{8}$	15	24	35	
	2			0	5	$\boxed{12}$	21	32	
	3				0	7	$\boxed{16}$	27	
	4					0	9	$\boxed{20}$	multiples of 4
	5						0	11	odd numbers
	6							0	

Explain Aakar's table.

Can you see any more diagonal patterns in the table? Write each pattern in terms of x.

How can Aakar use the number patterns to explain why some integers cannot be expressed as the difference of two squares?

Sample Responses to Discuss: Heng

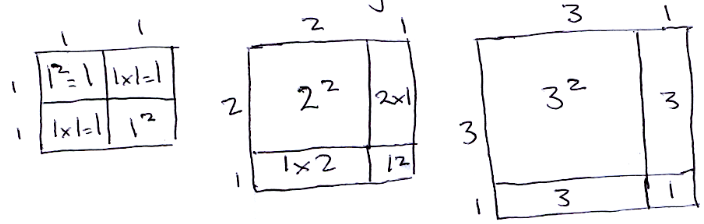
I am going to look at consecutive square numbers first: ${}^{\llcorner}$diff. of 1

$2^2 - 1^2 = 4 - 1 = 3$

$3^2 - 2^2 = 9 - 4 = 5$

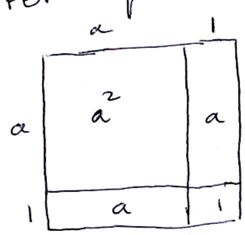
$4^2 - 3^2 = 16 - 9 = 7$ All odd numbers

To show these using areas:



$4^2 - 3^2 = 3 + 3 + 1$
$= 7$

For squares of length a and a+1



$(a+1)^2 - a^2 = a + a + 1$
$= 2a + 1$

a can be odd or even, but 2a is always even, so 2a+1 is always odd

What is unclear about Heng's area diagrams?

.....

.....

.....

.....

.....

What further work could Heng do?

.....

.....

.....

.....

.....

How Did You Work?

1. Compare the sample responses and your group response. What are the advantages and disadvantages of each approach?

	Advantages	Disadvantages
Aakar		
Heng		
Our group work		

2. In our group work we made some assumptions **Yes/ No**

These assumptions were ....

....

....

3. What could you do to improve your group work? Please explain your reasons.

....

....

4. What difficulties do you think someone new to the task will face?

....

....

The Difference of Two Squares

$5^2 - 3^2$

Some integers can be written as the difference of two square integers.

For example:

9 can be written as $5^2 - 4^2$

$15^2 - 7^2$

$10^2 - 7^2$

$100^2 - 99^2$

Which other integers can be expressed as the difference of two squares?

Planning a Joint Method

1. Take turns to explain your work and how you think it could be improved.

2. Listen carefully to each other and ask questions if you don't understand or agree.

3. Once everyone in the group has explained their method, plan a joint method that is better than each of your separate ideas.

4. Make sure that everyone in the group can explain the reasons for your chosen method.

5. Write a brief outline of your planned method on your large sheet of paper.

Sample Responses to Discuss

1. Each choose a sample response to work through and write your answers on your mini-whiteboards.

2. Explain your answers to the rest of the group.

3. Listen carefully to explanations. Ask questions if you don't understand.

4. Once everyone is satisfied with the explanations, write the answers to the questions below the sample response.

5. Try to ensure that the student who writes the answers is not the student who explained them.

Sample Responses to Discuss: Aakar

I will start by constructing a table so that I can check I have looked at the integers produced by all possible combinations.

Difference of two squares : $x^2 - y^2$

x

y

x^2-y^2	0	1	2	3	4	5	6
0	0	1	$\boxed{4}$	9	16	25	36
1		0	3	$\boxed{8}$	15	24	35
2			0	5	$\boxed{12}$	21	32
3				0	7	$\boxed{16}$	27
4					0	9	$\boxed{20}$
5						0	11
6							0

multiples of 4

odd numbers

Sample Responses to Discuss: Heng

I am going to look at consecutive square numbers first:

diff. of 1

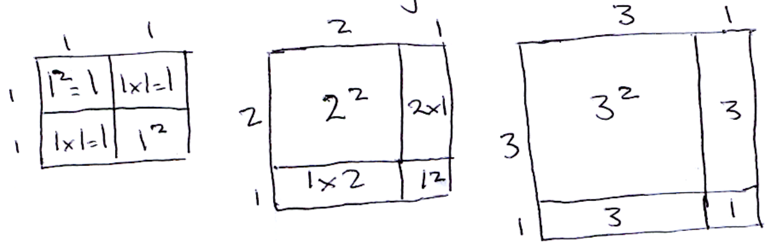
$2^2 - 1^2 = 4 - 1 = 3$

$3^2 - 2^2 = 9 - 4 = 5$

$4^2 - 3^2 = 16 - 9 = 7$

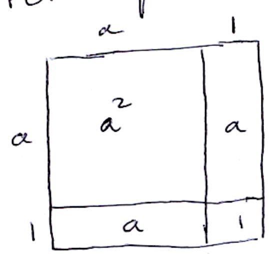
All odd numbers

To show these using areas:



$4^2 - 3^2 = 3 + 3 + 1$
$= 7$

For squares of length a and a+1



$(a+1)^2 - a^2 = a + a + 1$
$= 2a + 1$

a can be odd or even, but 2a is always even, so 2a+1 is always odd

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
Shell Center Team at the Center for Research in Mathematical Education
University of Nottingham, England:

Malcolm Swan,
Nichola Clarke, Clare Dawson, Sheila Evans, Colin Foster, and **Marie Joubert**
with
Hugh Burkhardt, Rita Crust, Andy Noyes, and **Daniel Pead**

We are grateful to the many teachers and students, in the UK and the US,
who took part in the classroom trials that played a critical role in developing these materials

The classroom observation teams in the US were led by
David Foster, Mary Bouck, and **Diane Schaefer**

This project was conceived and directed for
The Mathematics Assessment Resource Service (MARS) by
Alan Schoenfeld at the University of California, Berkeley, and
Hugh Burkhardt, Daniel Pead, and **Malcolm Swan** at the University of Nottingham

Thanks also to Mat Crosier, Anne Floyde, Michael Galan, Judith Mills, Nick Orchard, and Alvaro Villanueva who contributed to the design and production of these materials

This development would not have been possible without the support of
Bill & Melinda Gates Foundation

We are particularly grateful to
Carina Wong, Melissa Chabran, and **Jamie McKee**

The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org