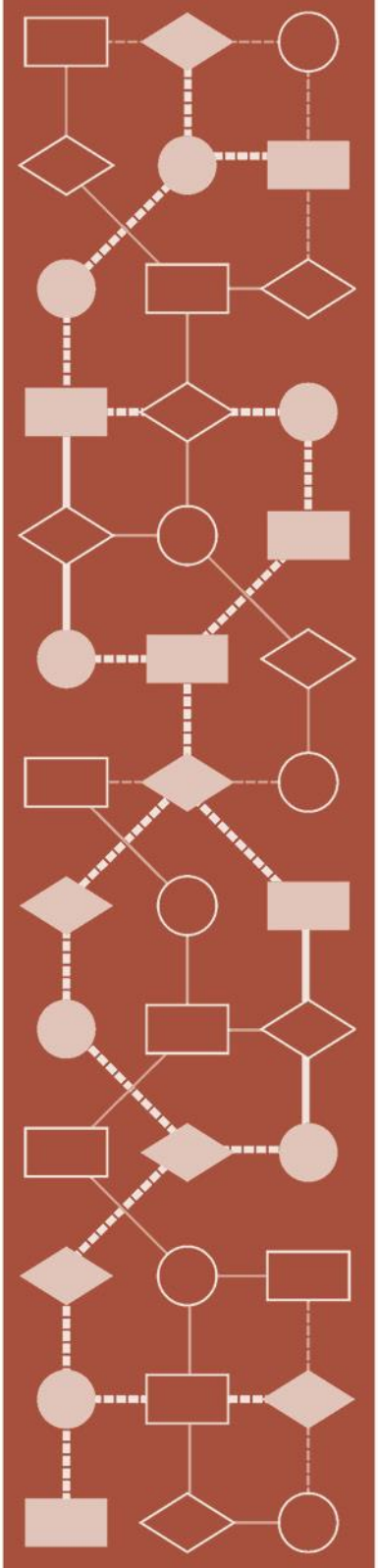
PROBLEM SOLVING



Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Comparing Value for Money: *Baseball Jerseys*

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Comparing Value for Money: *Baseball Jerseys*

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Interpret a situation and represent the variables mathematically. Select appropriate mathematical methods to use. Explore the effects of systematically varying the constraints.
- Interpret and evaluate the data generated and identify the break-even point, checking it for confirmation. Communicate their reasoning clearly.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 1, 2, 3, and 4:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
8. Look for and express regularity in repeated reasoning.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content*:

8.EE: Understand the connections between proportional relationships, lines, and linear equations.
Analyze and solve linear equations and pairs of simultaneous linear equations.
8.F: Use functions to model relationships between quantities.

INTRODUCTION

- Before the lesson students attempt the task individually. After reviewing responses, you formulate questions for students to consider when planning improvements to their work.
- At the start of the lesson, students think individually about the questions posed.
- Next, students work in small groups to combine their thinking and work together to produce a collaborative solution to the *Baseball Jerseys* task, in the form of a poster.
- In the same small groups, students evaluate and comment on sample responses, identifying the strengths and weaknesses in these responses and comparing them with their own work.
- In a whole-class discussion students compare and evaluate the strategies they have seen and used.
- In a follow-up lesson, students reflect for 10 minutes on their work and what they have learned.

MATERIALS REQUIRED

- Each individual student will need a copy of the assessment task: *Baseball Jerseys,* some plain paper, a mini-whiteboard, pen, and eraser, and a copy of the *How Did You Work?* questionnaire.
- Each small group of students will need a large sheet of paper, some felt tipped pens, and copies of *Sample Responses to Discuss*.
- Provide calculators and graph paper only when requested.
- There is a projector resource to support whole-class discussions.

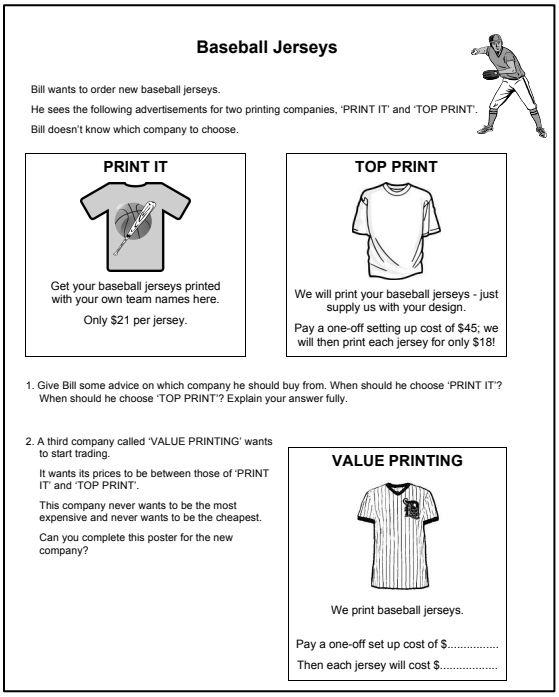
TIME NEEDED

15 minutes before the lesson, a 70-minute lesson (or split into two shorter ones), and 10 minutes in a follow-up lesson. Timings given are only approximate.

BEFORE THE LESSON

Assessment task: *Baseball Jerseys* (15 minutes)

Have the students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.



Baseball Jerseys

Bill wants to order new baseball jerseys.
He sees the following advertisements for two printing companies, 'PRINT IT' and 'TOP PRINT'.
Bill doesn't know which company to choose.

PRINT IT

Get your baseball jerseys printed with your own team names here.
Only $21 per jersey.

TOP PRINT

We will print your baseball jerseys - just supply us with your design.
Pay a one-off setting up cost of $45; we will then print each jersey for only $18!

1. Give Bill some advice on which company he should buy from. When should he choose 'PRINT IT'? When should he choose 'TOP PRINT'? Explain your answer fully.

2. A third company called 'VALUE PRINTING' wants to start trading.
It wants its prices to be between those of 'PRINT IT' and 'TOP PRINT'.
This company never wants to be the most expensive and never wants to be the cheapest.
Can you complete this poster for the new company?

VALUE PRINTING

We print baseball jerseys.
Pay a one-off set up cost of $.....
Then each jersey will cost $.....

Give each student a copy of the assessment task *Baseball Jerseys* and some plain paper for them to work on. Provide calculators if requested.

> *Read through the questions and try to answer them as carefully as you can. Show all your working, so that I can understand your reasoning.*
>
> *As well as trying to solve the problem, I want you to see if you can present your work in an organized and clear way.*

It is important that, as far as possible, students are allowed to answer the questions without your assistance.

Students who sit together often produce similar answers then, when they come to compare their work, they have little to discuss. For this reason, we suggest that, when students do the task individually, you ask them to move to different seats. At the beginning of the formative assessment lesson allow them to return to their usual seats. Experience has shown that this produces more profitable discussions.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem solving approaches.

We suggest that you do not score students' work. Research shows that this will be counter-productive, as it will encourage students to compare their scores and will distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students at the beginning of the lesson.

Common issues:	Suggested questions and prompts:
Has difficulty getting started	• What do you know? • What do you need to find out? • What calculations could you do with the information you have?
Omits to use all given information For example: The student may not have taken into account the $45 setting up cost for 'Top Print'.	• Write in your own words the information given. • What numbers are fixed? What can vary?
Work is unsystematic For example: The student writes 'Print It' 5 and 'Top Print' 10; 10 cost more etc.	• Can you organize the costs of different numbers of jerseys made by the two companies in a systematic way? • What would be sensible values to try? Why? • How might you organize your work?
Makes incorrect assumptions For example: The student assumes Bill should always choose 'Print It' because they don't have a $45 setting up cost like 'Top Print'.	• Will 'Top Print' always be more expensive? • How much will it cost to print 20 shirts with each company?
Work is poorly presented For example: The student presents the work as a series of unexplained numbers and/or calculations. Or: The student draws a table without headings. Or: The student circles or underlines numbers and it is left to the reader to work out why this is the answer as opposed to any other calculation.	• Would someone unfamiliar with the task easily understand your work? • Have you explained how you arrived at your answer?
Has difficulties when using graphs/equations For example: The student inaccurately plots lines, does not label axes, or does not explain the purpose of the graph. Or: The student makes a mistake when solving the equation.	• Would someone unfamiliar with the task easily understand your work? • How can you check your answer? • How do your answers help you to solve the problem? • Does your answer seem sensible?
Considers a specific case for comparison For example: The student states there are nine players in a baseball team so finds the cost of nine jerseys from each of the two companies (Q1). Or: The student completes a set-up fee and cost per jersey for 'Value Printing' based on a specific number of jerseys and doesn't explore what happens for different numbers of jerseys (Q2).	• What if Bill wanted to buy more/less than nine jerseys? Who should he buy them from? • 'Value Printing' never wants to be the cheapest/most expensive company. You have shown this is the case for [specific number] jerseys, what would happen if you were buying less/more? Would this still be the case?
Correctly answers all the questions The student needs an extension task.	• Try to find a different solution for the pricing of 'Value Printing'. Is there a way of describing all possible solutions?

SUGGESTED LESSON OUTLINE

Reviewing individual solutions to the task (10 minutes)

Give each student a mini-whiteboard, a pen, and an eraser and return their work on the *Baseball Jerseys* task. You may want to show the class Slide P-1 of the projector resource.

If you have not added questions to individual pieces of student work, either give each student a printed version of your list of questions with the questions that relate to their work highlighted, or write your list of questions on the board so that students can select questions from the board that are appropriate to their own work.

Recall what we were working on previously. What was the task about?

I have had a look at your work and have some questions I would like you to think about.

On your own, carefully read through the questions I have written. I would like you to use the questions to help you to think about ways of improving your work.

Use your mini-whiteboards to make a note of anything you think will help to improve your work.

If mini-whiteboards are not available, students may want to use the back of their response to the task to jot down their ideas about ways to improve their work. This is an opportunity for students to review their own work before working collaboratively on producing a group solution. Whilst students are reviewing their work, it may be appropriate to ask individual students questions that help them to clarify their thinking.

Collaborative small-group work: preparing joint solutions on posters (20 minutes)

Organize the class into groups of two or three students.

Today you are going to work together in your group to produce a joint solution to the Baseball Jerseys task that is better than your individual work.

Before students have another go at the task, they need to discuss what they have learned from reviewing their individual solutions. This will enable them to decide which of their different approaches is better.

You each have your own individual solution to the task and have been thinking about how you might improve it, using the questions I have posed.

I want you to share your work with your partner(s). Take turns to explain how you did the task and how you think it could be improved.

If explanations are unclear, ask questions until everyone in the group understands the individual solutions.

To check that students understand what they are being asked to do, ask a student to explain it:

Steven, tell me what you are going to do in your groups.

Slide P-2 summarizes these instructions.

Once students have had chance to discuss their work, hand out a sheet of poster paper and some felt tipped pens to each group of students. Provide calculators and graph paper if students request them. Display Slide P-1 of the projector resource.

Having discussed the work you have done individually, in your group agree on the best method for completing the problem and produce a poster that shows a joint solution to the Baseball Jerseys task which is better than your individual work.

Make sure that you answer the task questions clearly and write explanations on your poster. Include any assumptions you have made on your poster.

Again check that students understand what they are being asked to do by asking someone to explain the task to the rest of the class.

While students are working in small groups you have two tasks: to note different student approaches to the task and to support student problem solving.

Note different student approaches

Note any common mistakes. For example, are students consistently using all the given information? Which math do they choose to use? How do they use it? Attend to the students' mathematical decisions. Do they track their progress in their use of their chosen mathematics? Do they notice if they have chosen a strategy that does not seem to be productive? If so, what do they do? You can use this information to focus a whole-class discussion towards the end of the lesson.

Support student problem solving

Try not to make suggestions that move students towards a particular approach to the task. Instead, ask questions that help them to clarify their thinking, promote further progress and encourage students to develop self-regulation and error detection skills.

You may find that some students find it difficult to keep more than one piece of information in mind at the same time. For example, you may ask them to consider these two questions:

If Bill were to order ten jerseys from 'Print It', how much would they cost?

If Bill were to order ten jerseys from 'Top Print', how much would they cost?

Students who organize their work into a table may struggle to select appropriate column headings or omit column headings for the 'Number of jerseys', 'Cost from Print It' and 'Cost from Top Print'.

How can you make your table clearer to someone who is unfamiliar with the task?

If students are struggling to produce a joint solution to the task, encourage them to identify the strengths and weaknesses of the methods employed in their individual responses. Can any of these methods be improved to produce a group solution that is better than the original individual responses? Can they think of any other approaches to try?

What have you done that you both [all] agree on?

What else do you need to find?

Have you used all the information given in the task?

What do you now know that you didn't know before?

Do your calculations make sense?

What assumptions have you made? Do you think they are reasonable?

You may also want to use some of the questions in the *Common issues* table to support your own questioning. The purpose of these questions is to help students to track and review their problem solving strategies. They should be encouraged to give reasons for the choices they have made.

Sharing different approaches (10 minutes)

Once groups have completed their posters, display them at the front of the room. Hold a whole-class discussion on the methods used to produce a group solution. Ask two groups of students to describe the method used and the ways in which this method differs to their initial individual responses. Did the students check their work? If they did, what checking method did they use?

Extending the lesson over two days

If you are taking two days to complete the unit then you may want to end the first lesson here. At the start of the second day, briefly remind students of their previous work before moving on to the next collaborative activity.

Collaborative analysis of *Sample Responses to Discuss* (20 minutes)

After students have had sufficient time to discuss some different approaches, distribute copies of the *Sample Responses to Discuss* to each group.

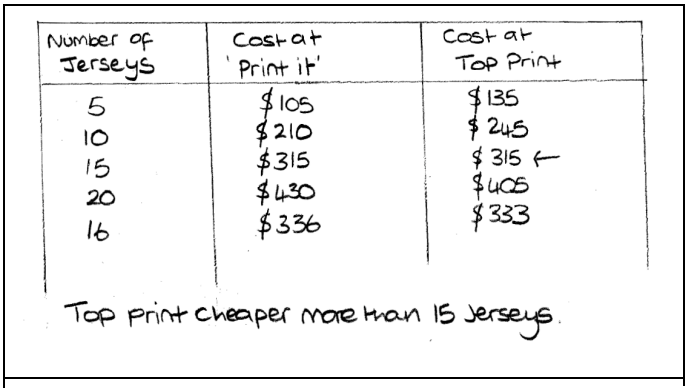
> *In your groups you are now going to look at some student work on the task. Notice in what ways this work is similar to yours and in which ways it is different.*
>
> *There are some questions for you to answer as you look at the work. You may want to add annotations to the work to make it easier to follow.*

This task gives students an opportunity to evaluate a variety of possible approaches to the task, without providing a complete solution strategy. Students should thoughtfully answer the questions below each piece of sample student work and be encouraged to think carefully about ways in which it could be improved.

It may not be appropriate, or there may not be enough time, for students to analyze all of the four sample responses. If students have been successful, then it may be better to issue sample responses that challenge them to approach the task using a different method. For example, students that have calculated the comparative prices for specific numbers of jerseys could be given Jeremiah's work, which shows the beginnings of an algebraic approach; students that have used an algebraic approach could be given Tanya's work that shows a graphical approach. Students that have struggled with a particular approach may benefit from seeing a student version of the same strategy.

During the small group work, support the students as before. Also, check to see which of the explanations students find more difficult to understand. Note similarities differences between the sample approaches and those the students took in the small group work.

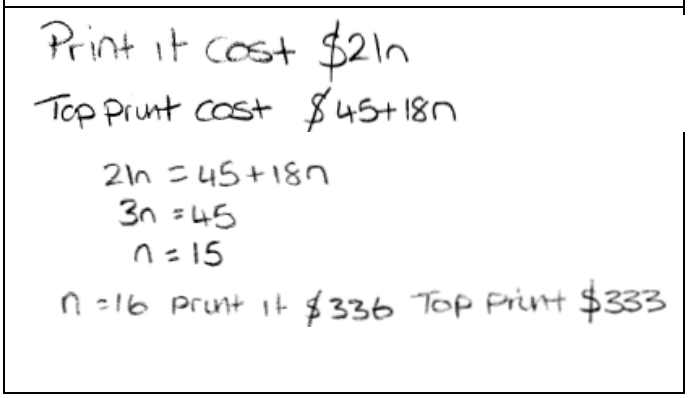
Danny has found an effective way to organize his work using a table, but has made some arithmetical mistakes: 10 jerseys at 'Top Print' should be $225 and 20 jerseys at 'Print It' should be $420. However, his reasoning is correct and these errors do not change his conclusions.



Number of Jerseys	Cost at 'Print it'	Cost at Top Print
5	$105	$135
10	$210	$245
15	$315	$315 ←
20	$430	$405
16	$336	$333

Top print cheaper more than 15 jerseys.

Jeremiah has tried an algebraic approach and found a correct solution for the break-even point. He checks the costs for more than 15 jerseys by calculating the cost of 16 jerseys from each of the two companies.



Print it cost $21n

Top Print cost $45+18n

$21n = 45 + 18n$

$3n = 45$

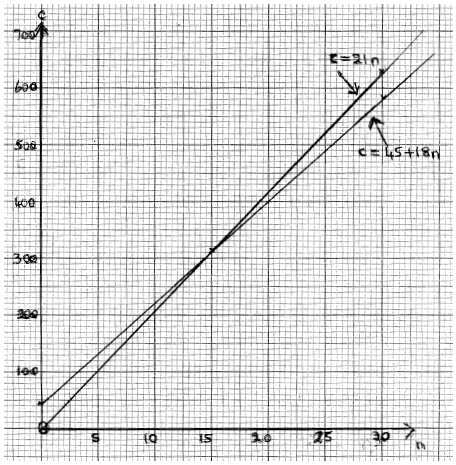
$n = 15$

n = 16 Print it $336 Top Print $333

Bella has tried a completely different approach. She reasons that once the setting up cost of $45 has been paid ($45 ÷ $3 = 15) 'Top Print' is $3 cheaper **per jersey**.

He would have to buy 15 jerseys to get the $45 set up cost gone. Then Top Print $3 cheaper.

Tanya writes equations for the costs of buying jerseys from the two companies: $c = 21n$ and $c = 45 + 18n$. She draws two line graphs representing the two equations but does not use an appropriate scale. She does not explain that the two graphs intersect at the point (15, 315), when the costs for the two companies are equal. The graph shows that for values of n less than 15, the cost for 'Print It' is less than 'Top Print'. For values of n greater than 15, the cost for 'Print It' is more than 'Top Print'.



Whole-class discussion (10 minutes)

Now hold a whole-class discussion to consider the different approaches seen in the sample responses and ask students to compare them with their own work.

Did any group use a similar method to Danny/Jeremiah/Bella/Tanya?

What was the same/different about the work?

What is unclear about Danny/Jeremiah/Bella/Tanya's work?

In what ways could the work be improved?

Did analyzing the responses enable you to see errors in your own work?

Of the four sample pieces of work, which do you think has the most complete solution? Which student has adopted the most appropriate approach?

As students compare the different solution methods, ask them to comment on their strengths and weaknesses. For example:

What are the advantages and disadvantages of Danny's work compared to Tanya's?

Students should be encouraged to consider the efficiency of different approaches and their appropriateness, as well as the accuracy and completeness of the sample responses. For example, when answering question two of the task a graphical approach provides a sensible strategy, but for question one it may not be considered necessary or advantageous. You may want to demonstrate this by showing students the graph on Slide P-7 of the projector resource. The scales for each axis have been changed, however the accuracy is still an issue.

Does this new scale help to identify the cost of jerseys?

For this problem, what are the advantages and disadvantages of using a graph?

You may also want to use Slides P-3, P-4, P-5, and P-6 of the projector resource and the questions in the *Common issues* table to support this whole-class discussion.

Follow-up lesson: individual reflection (10 minutes)

Once students have had a chance to discuss the sample responses as a whole-class, distribute the *How Did You Work?* questionnaire. Ask students to spend a couple of minutes, working individually, to answer the questions.

Think carefully about your work this lesson and the different methods you have seen and used.

On your own, answer the review questions as carefully as you can.

Some teachers give this as a homework task.

SOLUTIONS

1. For one or two jerseys, the one-off set-up cost means that each jersey is much more expensive from 'Top Print'. As the number of jerseys increases, the difference in the costs becomes smaller. For 15 jerseys the costs are equal and for more than 15 the cost at 'Top Print' becomes less than at 'Print It'.

Tabular method:

Number of jerseys	Cost at 'Print It'	Cost at 'Top Print'
1	\$21	\$63
2	\$42	\$81
5	\$105	\$135
10	\$210	\$225
15	\$315	\$315
20	\$420	\$405
25	\$525	\$495
16	\$336	\$333

Algebraic method:

'Top Print' jerseys cost less than 'Print It' jerseys when:

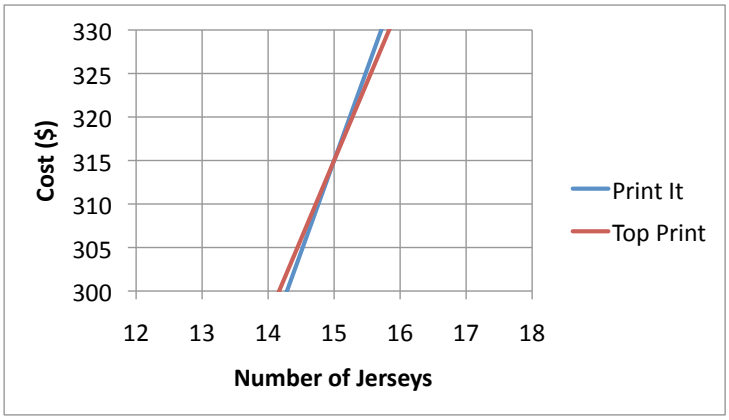
$18n + 45 \leq 21n$ n is the number of jerseys.

$3n \geq 45$

$n \geq 15$

Graphical method:

By plotting the lines C = 18n + 45 and C = 21n where C is the cost of jerseys and n is the number of jerseys it can be seen that the two lines intersect at the point (15, 315). This shows that when buying 15 jerseys it costs the same from both companies.



The advice to Bill is that if he is buying less than 15 jerseys then he should buy from 'Print It' as they are cheaper. If he is buying 15 jerseys then he can choose either company. If he is buying more than 15 jerseys then he should buy from 'Top Print'.

Students may refer to the size of roster or the number of players on the field and assume that Bill will be buying enough jerseys for each player. Any assumptions that are made should be clearly stated and explained.

2. There are multiple solutions to this question and students should be encouraged to extend their work to finding more than one solution. A possible solution would be:

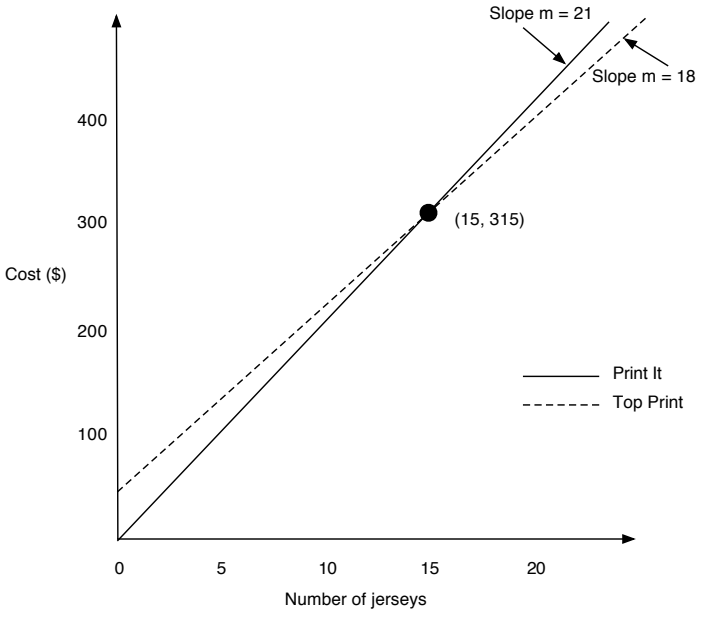
VALUE PRINTING

We print baseball jerseys.

Pay a one-off set up cost of $15

Then each jersey will cost $20

When thinking about this graphically, the line for the third company would need to pass through the point (15, 315) and have a slope between that of the other two lines i.e. $18 < m < 21$:



The third company will need to charge a one-off set-up cost, which is less than $45.

Baseball Jerseys

Bill wants to order new jerseys for his baseball team.

He sees the following advertisements for two printing companies, 'PRINT IT' and 'TOP PRINT'.

Bill doesn't know which company to choose.

PRINT IT

Get your baseball jerseys printed with your own team names here.

Only $21 per jersey.

TOP PRINT

We will print your baseball jerseys - just supply us with your design.

Pay a one-off setting up cost of $45; we will then print each jersey for only $18!

1. Give Bill some advice on which company he should buy from. When should he choose 'PRINT IT'? When should he choose 'TOP PRINT'? Explain your answer fully.

2. A third company called 'VALUE PRINTING' wants to start trading.

   It wants its prices to be between those of 'PRINT IT' and 'TOP PRINT'.

   This company never wants to be the most expensive and never wants to be the cheapest.

   Can you complete this poster for the new company?

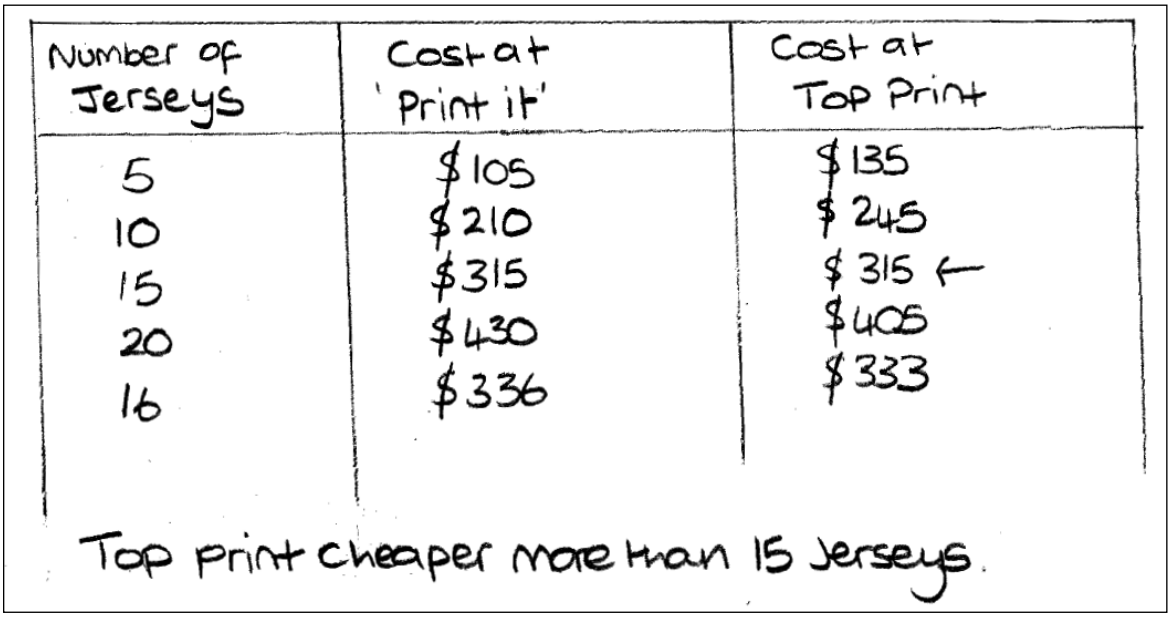
VALUE PRINTING

We print baseball jerseys.

Pay a one-off set up cost of $.....

Then each jersey will cost $.....

Sample Responses to Discuss: Danny



Number of Jerseys	Cost at 'Print it'	Cost at Top Print
5	$105	$135
10	$210	$245
15	$315	$315 ←
20	$430	$405
16	$336	$333

Top print cheaper more than 15 jerseys.

What mistakes has Danny made?

In what ways would Danny's mistakes affect his advice to Bill?

Why has Danny calculated the cost of 16 jerseys?

How could Danny's work be improved?

Sample Responses to Discuss: Jeremiah

Where does Jeremiah get the equation '$21n = 45 + 18n$' from?

Is Jeremiah's solution complete?

In what ways could Jeremiah's work be improved?

To help you to understand Jeremiah's work, what question(s) could you ask him?

Sample Responses to Discuss: Bella

He would have to buy 15 jerseys to get the $45 set up cost gone. Then Top Print $3 cheaper.

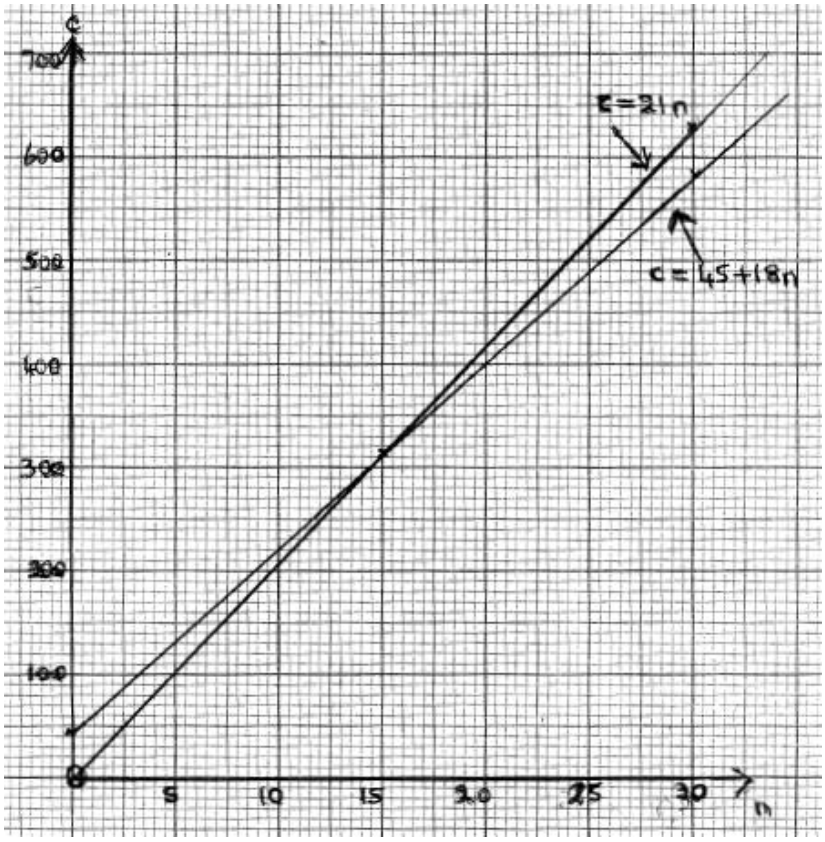
Is Bella's reasoning correct? Explain your answer.

What does Bella mean by '$3 cheaper'?

In what ways could Bella's work be improved?

To help you to understand Bella's work, what question(s) could you ask her?

Sample Responses to Discuss: Tanya



What is not clear about Tanya's work?

In what ways could Tanya's work be improved?

To help you to understand Tanya's work, what question(s) could you ask her?

How Did You Work?

Mark the boxes and complete the sentences that apply to your work.

1 Our group solution was mathematically better than my individual solution ☐ **OR** My individual solution was mathematically better than our group solution ☐

This is because .....

.....

2 We checked our method ☐

We checked our method by

.....

.....

We could check our method by

.....

.....

3 Our solution is similar to one of the sample responses ☐ OR Our solution is different from **all** the sample responses ☐

Our solution is similar to _____ *Add name of sample response*

I prefer **our method** / **the sample response** method (*circle)*

This is because .....

.....

.....

This is because .....

.....

.....

4 The solution most useful to Bill is _____ This is because .....

.....

.....

Baseball Jerseys

PRINT IT

Get your baseball jerseys printed with your own team names here.

Only $21 per jersey.

TOP PRINT

We will print your baseball jerseys - just supply us with your design.

Pay a one-off setting up cost of $45; we will then print each jersey for only $18!

VALUE PRINTING

We print baseball jerseys.

Pay a one-off set up cost of $.....

Then each jersey will cost $.....

1. Give Bill some advice. When should he choose PRINT IT? When should he choose TOP PRINT?

2. VALUE PRINTING never wants to be the most expensive and never wants to be the cheapest.

Sharing Individual Solutions

1. Take turns to share your work.

2. Describe how you did the task and how you think it could be improved.

3. If explanations are unclear, ask questions until everyone in the group understands the individual solutions.

Sample Responses to Discuss: Danny

Number of Jerseys	Cost at 'Print it'	Cost at Top Print
5	$105	$135
10	$210	$245
15	$315	$315 ←
20	$430	$405
16	$336	$333

Top print cheaper more than 15 jerseys.

Sample Responses to Discuss: Jeremiah

Print it cost $21n

Top Print cost $45+18n

$21n = 45 + 18n$

$3n = 45$

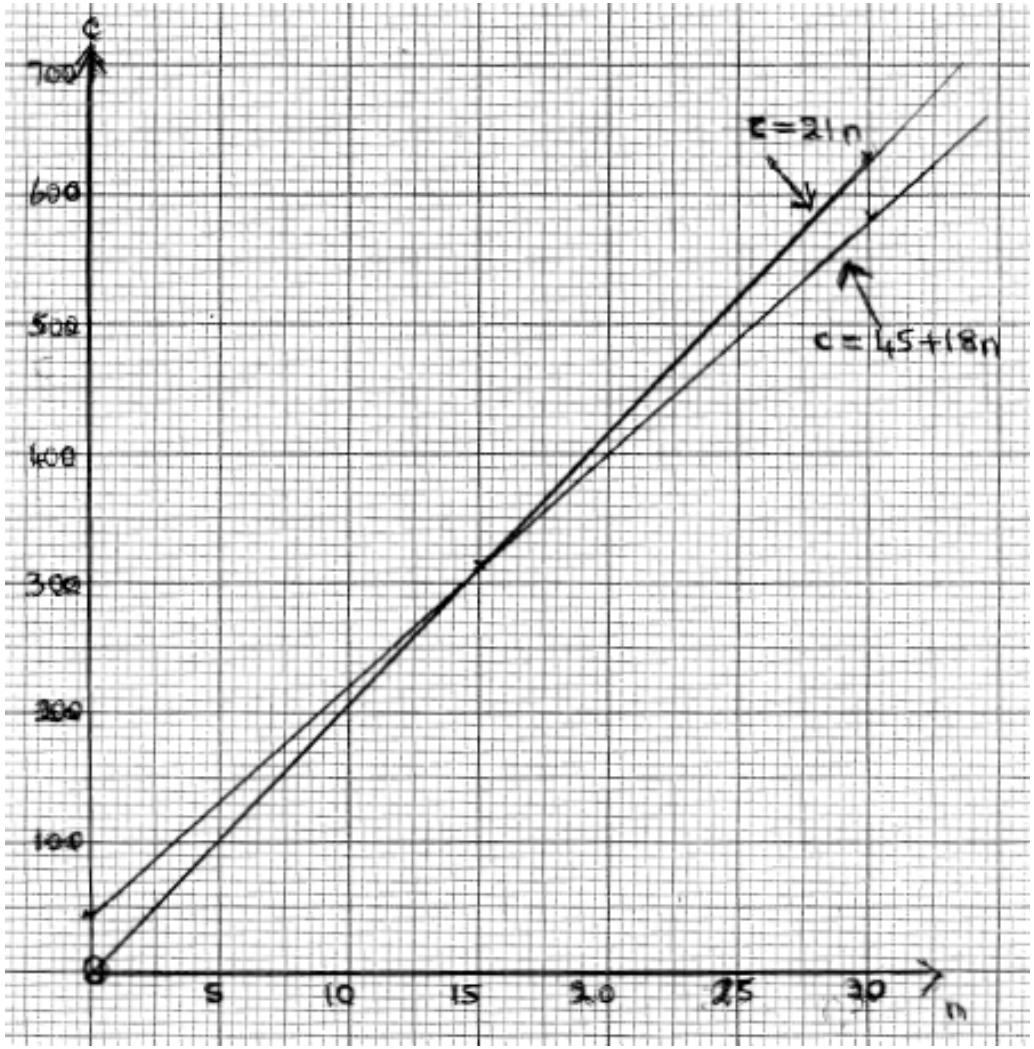
$n = 15$

n = 16 Print it $336 Top Print $333

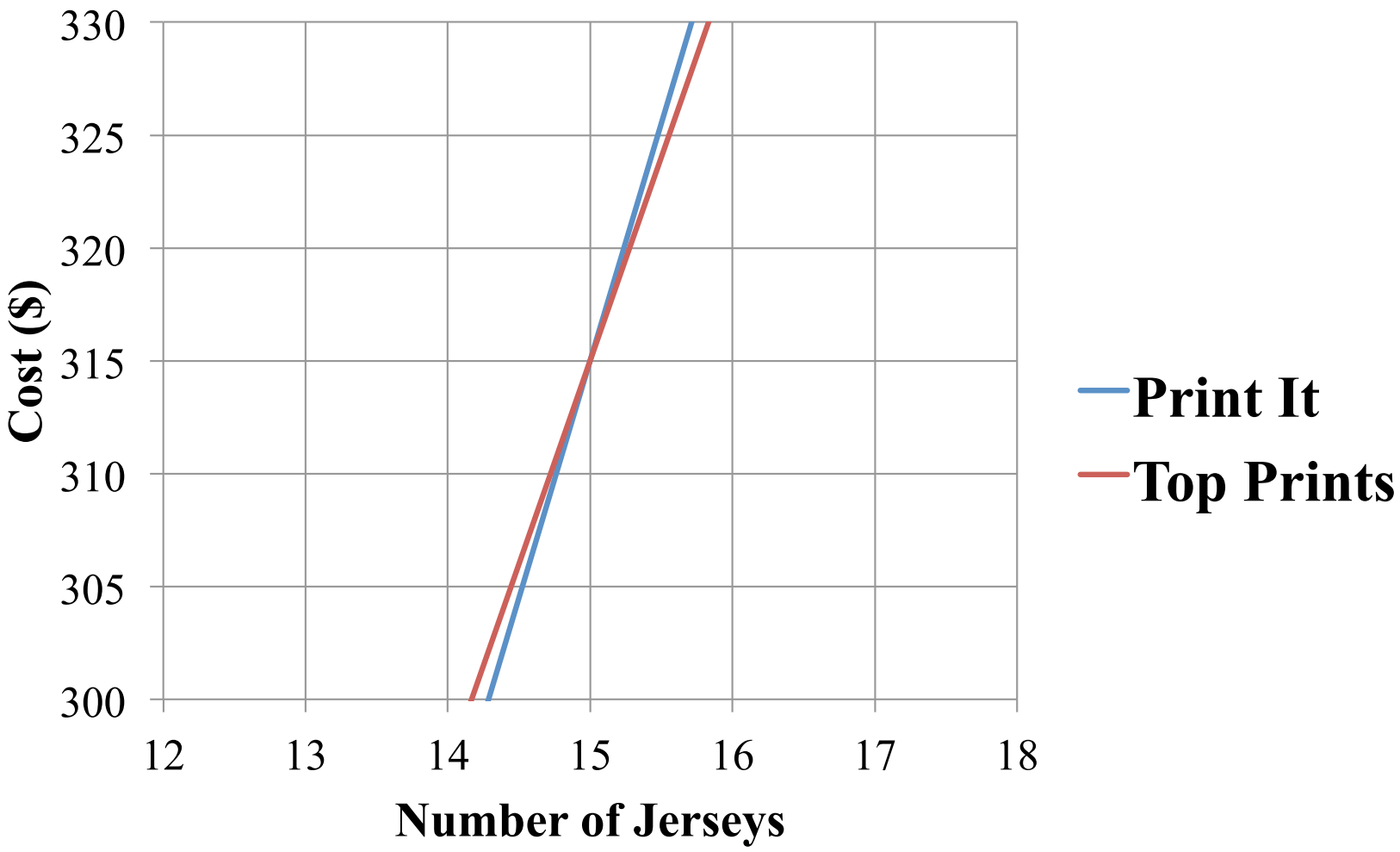
Sample Responses to Discuss: Bella

He would have to buy 15 jerseys to get the $45 set up cost gone. Then Top Print $3 cheaper.

Sample Responses to Discuss: Tanya



Graph Showing Cost of Jerseys



Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
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with
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