CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Finding Areas of Circles

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Finding Areas of Circles

MATHEMATICAL GOALS

This lesson unit is intended to help students to apply their knowledge of the relationship between the radius and the area of a circle. In particular, students will:

- Sketch and construct loci involving arcs of circles.
- Calculate the area of a circle, given its radius.
- Calculate the radius of a circle, given its area.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

7.EE: Solve real-life and mathematical problems using numerical and algebraic expressions and equations

7.G: Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

This lesson also relates to **all** the *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 2, 3, 4, and 6:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

The lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task designed to reveal their current understanding and difficulties. You then review their solutions and create questions for students to help them to improve their work. They receive this feedback in the follow-up lesson.
- After a whole-class introduction students work in small groups on a collaborative task, finding the rope length necessary to enable a goat to eat a specified area of grass.
- In the same small groups, students evaluate and comment on sample responses, identifying strengths and weaknesses and comparing them with their own work. After a whole-class discussion, they then try to improve their collaborative work.
- In an optional final whole-class discussion, students discuss the methods they have seen and used.
- Finally, students work alone on a new assessment task, or try to improve their original responses.

MATERIALS REQUIRED

- Each individual student will need a copy of the assessment tasks *The Goat and the Grass* and *The Goat and the Grass (revisited)*. Calculators, rulers and compasses should be made available.
- Each small group will need plain paper and copies of the *Field* and *Sample Responses to Discuss*.

TIME NEEDED

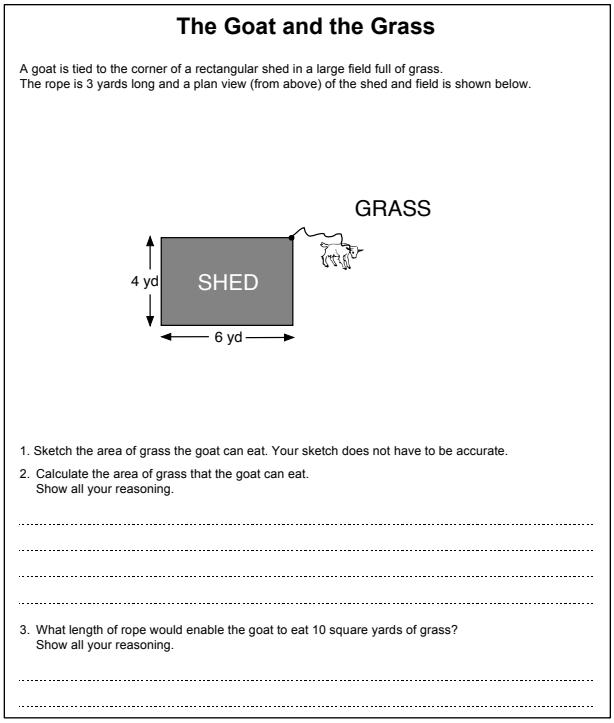
15 minutes before the lesson, a 100-minute lesson (or two 60-minute lessons), and 15 minutes in a follow-up lesson. Timings are approximate and will depend on the needs of the class.

BEFORE THE LESSON

Establish what the students already know about the area of a circle. They should have been previously taught how to calculate the area of a circle before they attempt this lesson. If you are unsure of this, ask them to calculate the area of a circle with given radius in addition to the assessment task below.

Assessment task: *The Goat and the Grass* (15 minutes)

Have the students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.



Give each student a copy of *The Goat and the Grass*. Calculators and compasses should be made available only on request. We want students to realize for themselves that the locus will be circular.

Introduce the task briefly and help the class to understand what they are being asked to do.

> *Can you see what the diagram is showing? We're looking down from above. [OK, so the goat isn't really lying on its side! Please excuse this.]*
>
> *You need to show everywhere the goat can reach and then try to calculate the area of grass it can eat.*
>
> *Have a go at the questions on the sheet. If you need more space, continue on the back.*

It is important that, as far as possible, students are allowed to answer the questions without your assistance.

Students should not worry too much if they cannot understand or do everything, because in the next lesson they will engage in a similar task, which should help them. Explain to students that by the end of the next lesson, they should expect to be able to answer questions like these confidently. This is their goal.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem-solving approaches.

We suggest that you do not score students' work. Research suggests that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students in the follow-up lesson.

Common issues:	Suggested questions and prompts:
Sketches a non-circular locus for the goat's extreme positions For example: The student draws a square, rectangle or triangle.	• If the goat was tethered to a post in an empty field, how far could it go in any direction? • Think about where the goat can go if the rope is kept taut. What path will it trace out? • What shape best describes the area of grass that the goat can reach?
Sketches an incorrect circular locus for the goat's extreme positions For example: The student draws a circular arc with a radius greater than 3 yards.	• How long is the rope in your drawing?
Ignores the constraint of the shed For example: The student calculates the area of a circle of radius 3 yards.	• How much of the circle contains grass?
Involves the area of the shed For example: The student calculates the area of a circle of radius 3 yards and then subtracts the area of the shed.	• Can you explain why you subtracted the area of the shed?
Fails to carry out a reverse calculation (Q3) For example: The student calculates the area of a circle of radius 10 yards, instead of the radius of a circle with area 10 square yards.	• Look at Q3 again carefully. What are you told and what do you need to find?
Uses an inefficient method For example: The student uses a lengthy trial and improvement or measurement approach (Q3).	• Can you think of a way to calculate the length of rope directly? • What math operations were involved when you knew the length of the rope and were asked to find the area of the grass? • If you know the area of the shape, how can you work backwards to find the length of the rope?
Answers all questions correctly and with full explanation	• Can you work out what length of rope would enable the goat to eat 50 square yards of grass?

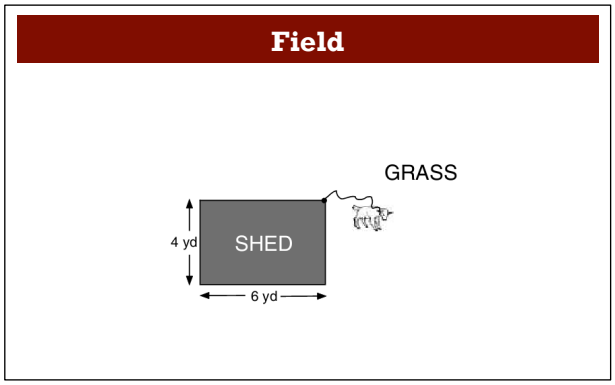
SUGGESTED LESSON OUTLINE

Whole-class interactive introduction (15 minutes)

Remind the class of the assessment task they have already attempted.

Do you remember the task about the goat in the field? What did you have to work out?

Display Slide P-1 of the projector resource showing the goat and the shed:



If students have not completed the assessment task correctly, do not go over this now and correct their errors. It is better to allow their difficulties to be addressed during today's lesson.

What assumptions did you make to answer the question? What things did you have to assume?

Students might mention that they assumed that the rope is securely attached to a fixed point on the shed, does not stretch, does not break and that the field is very large, flat and covered uniformly in grass. The rope attaches around the goat's neck, but the goat eats with its mouth, and can bend its neck to reach the ground, so for simplicity we treat the goat as a 'point' and assume that the rope is horizontal (making the problem two-dimensional). We assume that the goat explores right to the limits of where the rope will allow it to reach. There are many other things that students might also comment on.

Do you think it was reasonable to make those assumptions? Why / Why not?

Being explicit about assumptions is an important element of mathematical modeling and by discussing this early on in the lesson it may prevent questions and confusion later.

Collaborative small-group work: *Field* (20 minutes)

Today we are going to work on a related problem to do with the goat and the field.

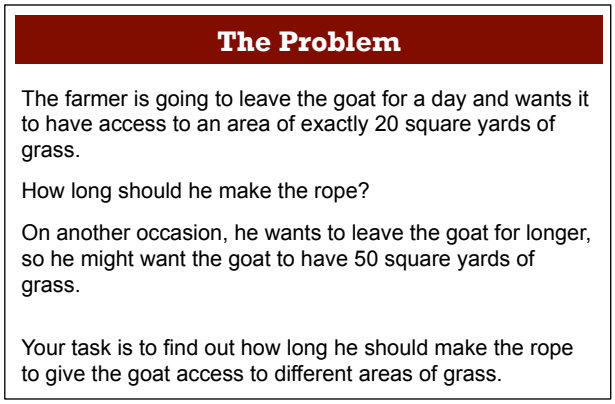
Suppose the farmer is going to leave the goat for a day and wants it to have access to exactly 20 square yards of grass. How long should he make the rope?

Students will not be able to answer this question immediately, although they might offer rough guesses at this stage, which you could accept without evaluation. Broaden their investigation by asking:

On another occasion he might need to leave the goat for longer, so he might want the goat to have 50 square yards of grass.

Your task is to find out how long he should make the rope to give the goat access to different areas of grass.

Display Slide P-2 of the projector resource, showing the details of the problem:



Ask students to work in groups of two or three.

Give each group some blank paper and copies of the sheet called *Field*. Compasses and rulers should be available upon request; avoid issuing equipment in an obvious way, as it may push students towards drawing circles and measuring accurately rather than allowing them to think about the task in their own ways.

You have two tasks during the group work: to make a note of student approaches to the task and to support students working as a group.

Make a note of student approaches to the task

Listen and watch students carefully. In particular, notice how students make a start on the task, where they get stuck and how they overcome any difficulties.

Do they begin with a trial length for the rope and see how much grass area it allows, or do they try to work backwards directly from the area to the length of rope? Do students realize that they will need a rope longer than 4 yards for an area of 50 yards and that this will therefore snag at the bottom right corner of the shed? How do they deal with this? Do they make careful sketches or are they more focused on the numbers? Are their answers realistic? Do they relate their answers back to the problem?

You might like to ask:

What do you already know? What are you trying to find out?

What calculations could you do? How do your calculations relate to the problem?

What effect does having a longer rope have on the area of grass the goat can eat?

Support students working as a group

As students work on the task, support them in working together. Encourage them to take turns and if you notice that one partner is doing all the sketching or that they are not working collaboratively on the task, ask students in the group to explain what someone else in the group has done.

Possible extension

Students who have successfully tackled finding the length of rope required for 20 and 50 square yards of grass might like to consider the length of rope needed for the goat to have access to 100 square yards of grass. In this case the length of the rope will be longer than both dimensions of the shed so students will need to consider carefully what this area will look like when carrying out their calculations.

Collaborative analysis of *Sample Responses to Discuss* (25 minutes)

Once students have had sufficient time to discuss their joint solutions, give each group copies of the *Sample Responses to Discuss*.

> *Now you are going to look at some other students' work on this problem. They are thinking about the second part of the task –* ***how to get 50 square yards of grass****.*

It may not be appropriate, or there may not be enough time, for all students to analyze all three sample responses. Each response highlights different features of the solution, and so, depending on the progress already made on the task, it may be appropriate to issue different sample responses to different groups. For example, groups that have relied solely on trial and improvement might benefit from looking at Shifa's solution.

Alternatively, for students working in groups of three it might be helpful to allocate each student with one of the three sample responses and then getting students to re-group based on which sample response they have been allocated. They can then discuss that particular piece of work in detail in these larger groups before returning to their original groups of three to report back on what they have discussed.

Display and explain to students Slide P-3 of the projector resource:

Sample Responses to Discuss

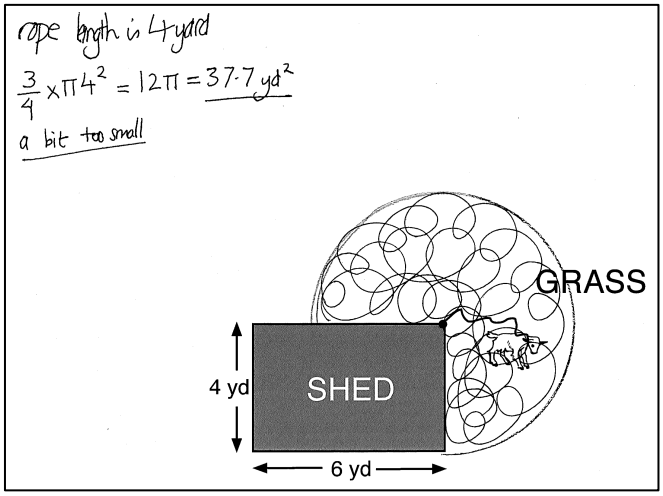
1. Take turns to explain to the rest of the group what the student in your sample response has done and what your answers are to the questions on the sheet.
2. Listen carefully to the explanations. Ask questions if you don't understand.
3. When everyone is satisfied with the explanations, you may want to change some of your own ideas!

This task gives students an opportunity to evaluate a variety of possible approaches to the task and to notice any differences and/or similarities with their own work.

Students should thoughtfully answer the questions below each piece of sample student work and be encouraged to think carefully about ways in which the work could be improved.

Kadu assumes a rope length of 4 yards and correctly calculates the area of ¾ of the resulting circle. He also comments that his answer is too small for the 50 square yards that he was presumably aiming for. This approach suggests the beginning of a trial and improvement process.

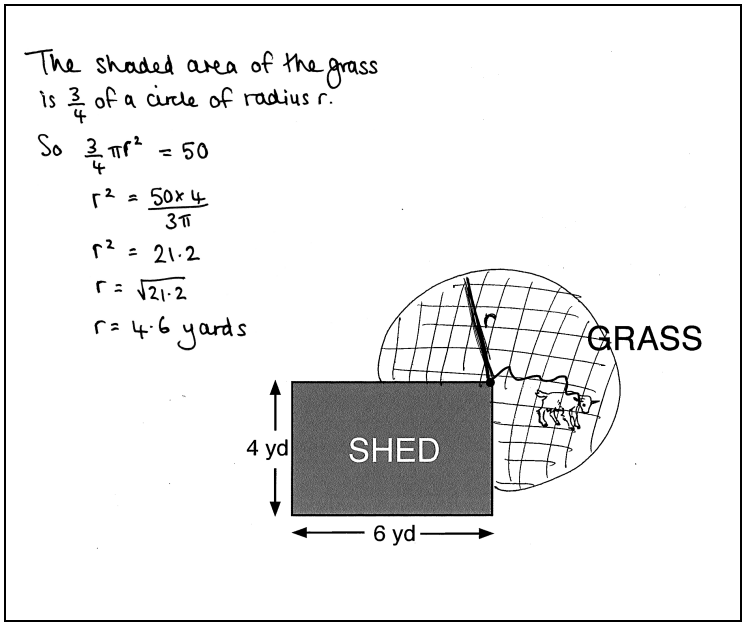
He seems to have stopped at this point, but students could extend his work by increasing the rope length. This will entail thinking about what happens at the bottom right corner for ropes longer than 4 yards.



Shifa begins with the area of 50 square yards and calculates the corresponding radius for the circle.

Students should notice that the value of 4.6 yards does not correspond to Shifa's diagram, where she assumes that *r* is less than 4 yards. It is an overestimate, since with this length of rope there would be some additional area below the shed, which she has not included.

Shifa's approach is included to suggest to students that an inverse algebraic approach might be more powerful than trial and improvement.

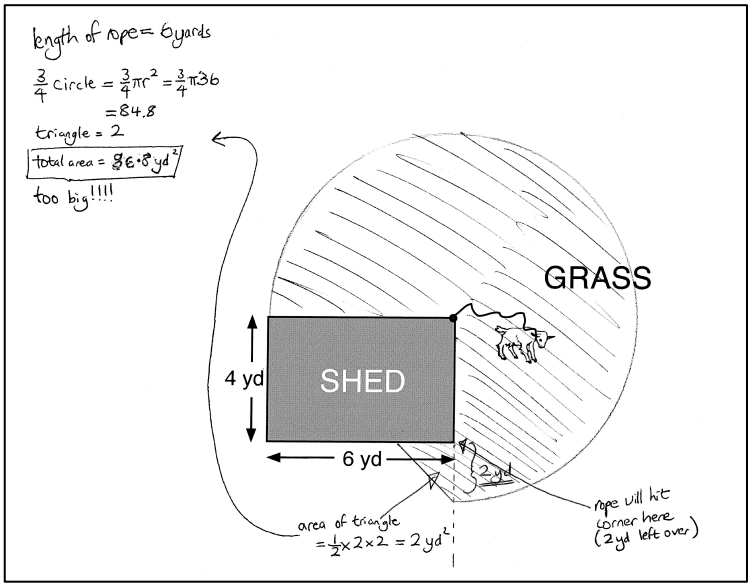


Alice tries a length of 6 yards, recognizing that there is a problem where the rope hits the bottom right corner of the shed.

She draws a triangle underneath the shed instead of a quadrant of radius 2 yards.

She correctly calculates the total area for her incorrect shape.

Alice's approach will suggest to students that they need to pay close attention to what happens when the rope is longer than one side of the shed.



Extending the lesson over two days

If you are taking two days to complete the unit then you may want to end the first lesson here. At the start of the second day, allow some time for students to remind themselves of the sample responses before discussing the different approaches as a whole-class.

Whole-class discussion: comparing different approaches (20 minutes)

Hold a whole-class discussion to consider the different approaches used within the sample responses. Look at each response in turn and ask students to comment on their strengths and weaknesses. It may be helpful to display Slides P-4 to P-6 during this discussion.

What did Kadu / Shifa / Alice do?

What is unclear about their work?

How is their work similar/different to what you did?

Once you have discussed each piece of work, ask students to compare and evaluate the different methods.

What do you think is the best way to tackle this problem? Why?

Students should be encouraged to consider the advantages and disadvantages of each method and whether the answers obtained are correct and reasonable.

Collaborative small-group work (20 minutes)

You have looked at the sample student work. Now your task is to use what you have seen there to improve your solution to the problem.

Use the suggested questions above to support students' learning and relate what they are doing to the sample student work.

Is your solution similar to one of the sample pieces of work? Which one? Why?

How is your work now benefiting from what you saw in the sample work?

Are there any particular mistakes that you are being careful to avoid now? What are they?

Whole-class discussion (optional)

You might wish to hold another whole-class discussion at this point, if you have time, to allow students to talk about the approaches they have taken and compare their answers. In particular, it is important to allow students to voice what they have noticed about different rope lengths and the effects this has on the area of grass available to the goat. Have students, for example, identified the borderline cases and been able to make some generalizations about the method for calculating the area between these points?

Follow-up lesson: reviewing the assessment task (15 minutes)

Give each student a copy of the assessment task *The Goat and the Grass (revisited)* and their original solutions to the assessment task *The Goat and the Grass*. If you have not added questions to individual pieces of work then write your list of questions on the board. Students then select from this list only those questions they think are appropriate to their own work.

Read through your papers from The Goat and the Grass and the questions [on the board/written on your paper.] Answer these questions and revise your response.

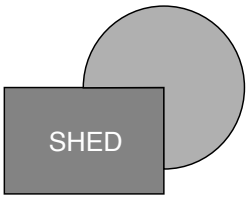
Now look at the new task sheet, The Goat and the Grass (revisited). Can you use what you have learned to answer these questions?

If students struggled with the original assessment task, you may feel it more appropriate for them to revisit *The Goat and the Grass* rather than attempting *The Goat and the Grass (revisited)*. If this is the case give them another copy of the original assessment task instead.

SOLUTIONS

Assessment task: *The Goat and the Grass*

1.



2. Area = $\frac{3}{4}\pi \times 3^2 = 21.2$ yd^2 (correct to 1 decimal place)

3. It is likely that students will use a trial and improvement method to calculate the length of the rope. Based on the solution to Q1 it is clear that the length of the rope will need to be less than 3 yards. Students should be encouraged to find an efficient way of calculating the length of rope given the area of grass available. The length of the rope can be calculated by recognizing that the area is given by:

$A = \frac{3}{4}\pi r^2 = 10$ square yards

$\Rightarrow r = \sqrt{\frac{40}{3\pi}}$ yards

$\Rightarrow r = 2.1$ yards (correct to 1 decimal place)

Collaborative task

When calculating the area (A square yards) from the length of the rope (r yards), the following formulae apply:

For $0 \le r \le 4$: $A = \frac{3}{4}\pi r^2$ (the area ranges from 0 to 12π yd^2)

For $4 \le r \le 6$: $A = \frac{3}{4}\pi r^2 + \frac{1}{4}\pi(r-4)^2$ (the area ranges from 12π to 28π yd^2)

For $6 \le r \le 10$: $A = \frac{3}{4}\pi r^2 + \frac{1}{4}\pi(r-4)^2 + \frac{1}{4}\pi(r-6)^2$ (the area ranges from 28π to 88π yd^2)

The values for the area of grass (20, 50 (and 100 square yards)) encourage students to consider what happens when the rope is shorter than the shortest side of the shed, when it is shorter than the longest side of the shed but longer than the shortest side (and when it is longer than both dimensions of the shed).

Students that struggled with the pre-assessment are likely to benefit from the opportunity to consider the most straightforward case where the length of the rope is shorter than the shortest side of the shed.

We might expect students to check out the areas of some of the borderline cases and then explain which one to use for different values of A. Some students might be able to sketch a graph relating r and A for different values of r, and this may be used to estimate a value for r that is needed for any given value of A.

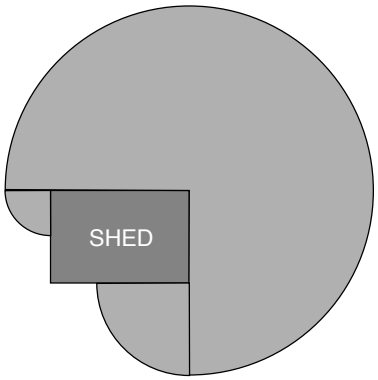
Students should be encouraged to be as efficient as possible in their solution strategies and look beyond repetitive trial and improvement techniques where possible.

The following table of values may be useful for checking student work:

Area (square yards)	Length of rope (yards) (correct to 2 decimal places)
10	2.06
20	2.91
30	3.57
40	4.12
50	4.59
60	5.01
70	5.39
80	5.74
90	6.06
100	6.37
110	6.65
120	6.92
130	7.17
140	7.41
150	7.64
160	7.86
170	8.07
180	8.28
190	8.48
200	8.67
210	8.86
220	9.04
230	9.22
240	9.40
250	9.57
260	9.73
270	9.90

Assessment task: *The Goat and the Grass (revisited)*

1.



2. Area = $\frac{3}{4}\pi \times 8^2 + \frac{1}{4}\pi \times 4^2 + \frac{1}{4}\pi \times 2^2 = 166.5 \text{ yd}^2$ (correct to 1 decimal place)

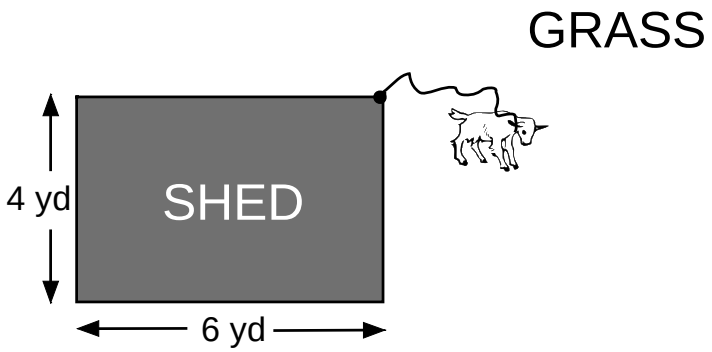
3. Students may well recognize that for an area of grass of 25 square yards the length of the rope will be shorter than the shortest side of the shed so the area is given by:

$A = \frac{3}{4}\pi r^2 = 25$ square yards

$\Rightarrow r = \sqrt{\frac{25 \times 4}{3 \times \pi}} = 3.3$ yards (correct to 1 decimal place)

The Goat and the Grass

A goat is tied to the corner of a rectangular shed in a large field full of grass.
The rope is 3 yards long and a plan view (from above) of the shed and field is shown below.

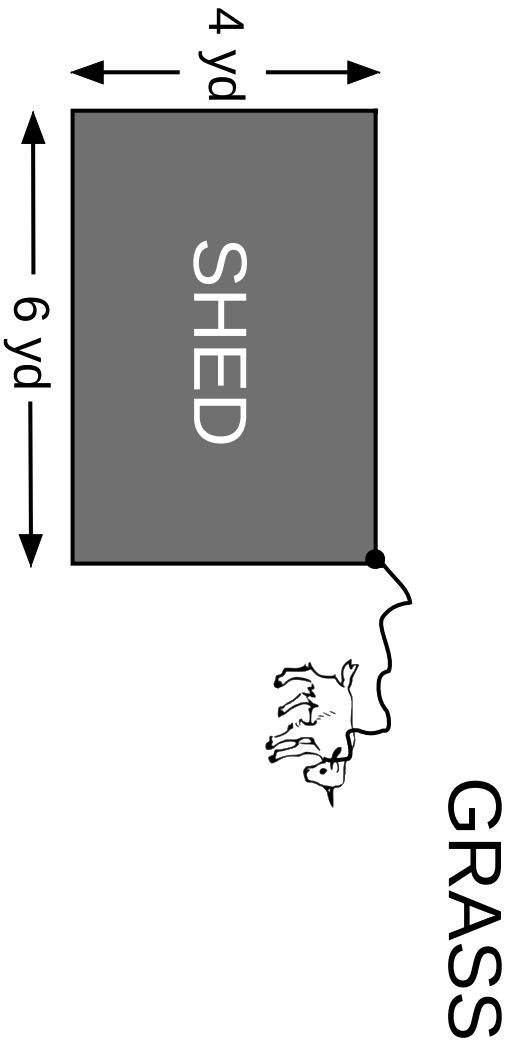


1. Sketch the area of grass the goat can eat. Your sketch does not have to be accurate.

2. Calculate the area of grass that the goat can eat.
   Show all your reasoning.

3. What length of rope would enable the goat to eat 10 square yards of grass?
   Show all your reasoning.

Field

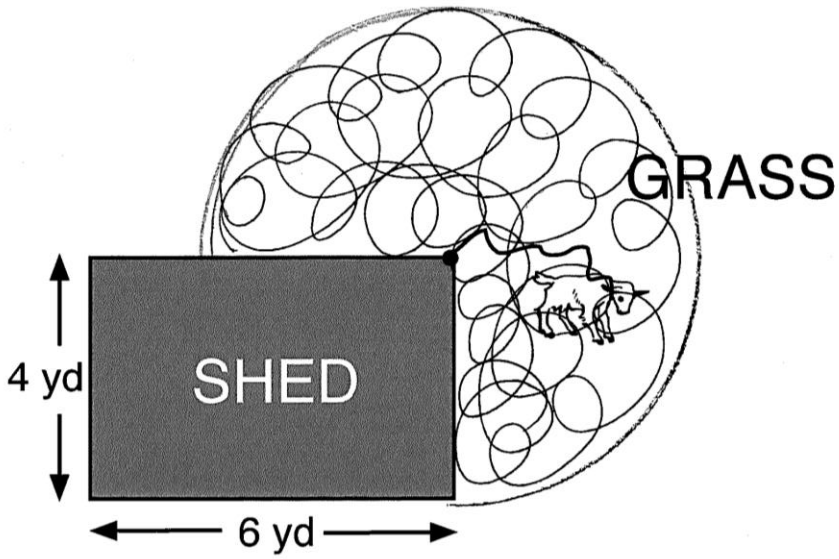


Sample Responses to Discuss: Kadu

rope length is 4 yard

$\frac{3}{4} \times \pi 4^2 = 12\pi = 37.7\ yd^2$

a bit too small



1. Try to explain what Kadu has done.

2. Do you agree with his conclusion? Why / Why not?

Sample Responses to Discuss: Shifa

The shaded area of the grass is $\frac{3}{4}$ of a circle of radius r.

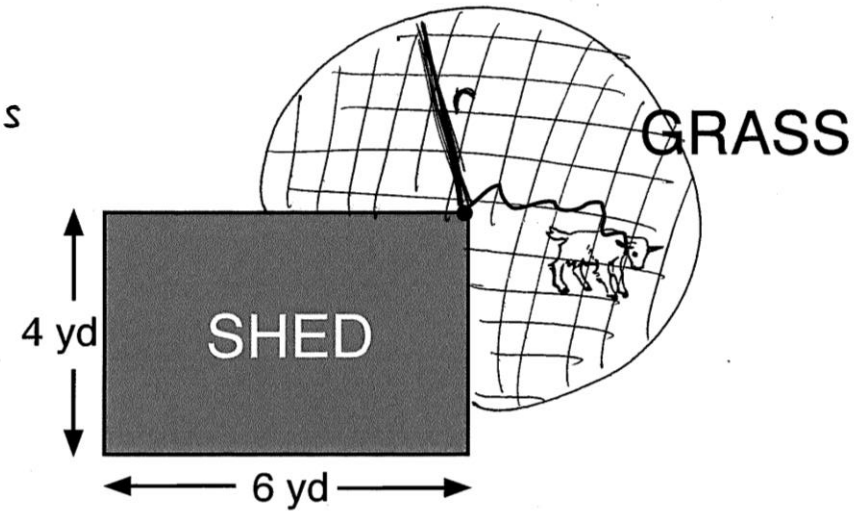
So $\frac{3}{4}\pi r^2 = 50$

$r^2 = \frac{50 \times 4}{3\pi}$

$r^2 = 21.2$

$r = \sqrt{21.2}$

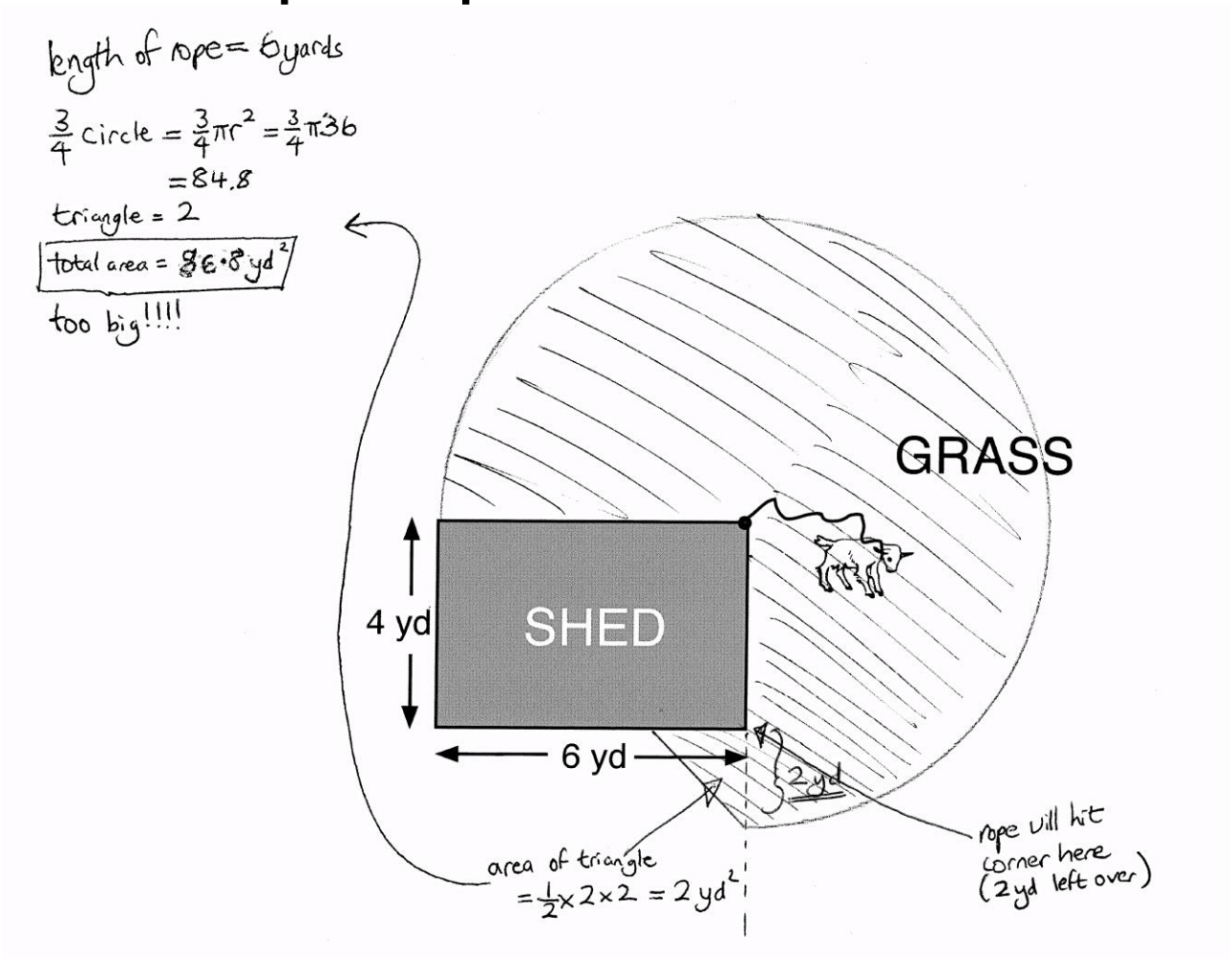
$r = 4.6$ yards



1. Try to explain what Shifa has done.

2. Do you agree with her conclusion? Why / Why not?

Sample Responses to Discuss: Alice

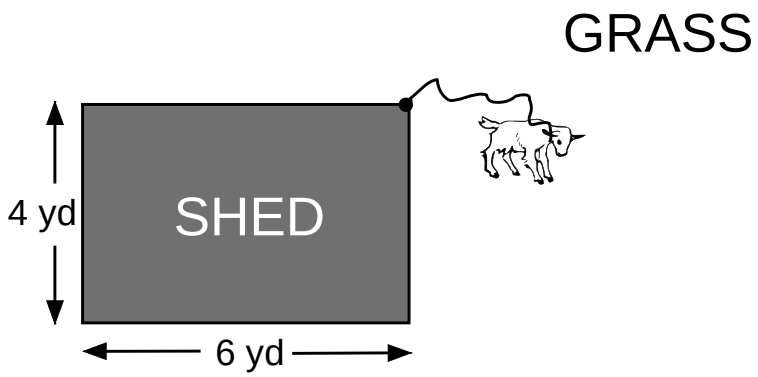


1. Try to explain what Alice has done.

2. Do you agree with her conclusion? Why / Why not?

The Goat and the Grass (revisited)

A goat is tied to the corner of a rectangular shed in a large field full of grass.
The rope is 8 yards long and a plan view (from above) of the shed and field is shown below.

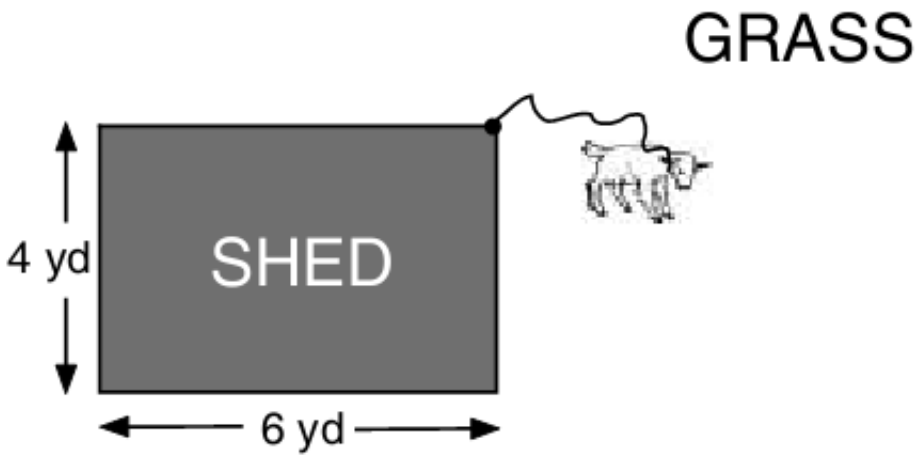


1. Sketch the area of grass the goat can eat. Your sketch does not have to be accurate.

2. Calculate the area of grass that the goat can eat.
   Show all your reasoning.

3. What length of rope would enable the goat to eat 25 square yards of grass?
   Show all your reasoning.

Field



The Problem

The farmer is going to leave the goat for a day and wants it to have access to an area of exactly 20 square yards of grass.

How long should he make the rope?

On another occasion, he wants to leave the goat for longer, so he might want the goat to have 50 square yards of grass.

Your task is to find out how long he should make the rope to give the goat access to different areas of grass.

Sample Responses to Discuss

1. Take turns to explain to the rest of the group what the student in your sample response has done and what your answers are to the questions on the sheet.

2. Listen carefully to the explanations.
   Ask questions if you don't understand.

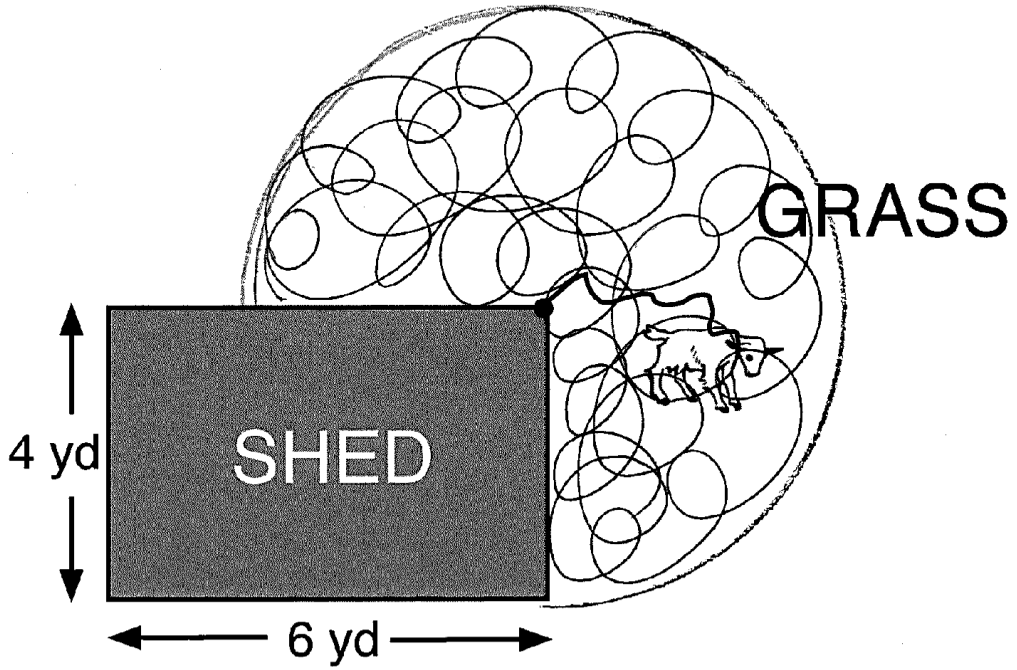
3. When everyone is satisfied with the explanations, you may want to change some of your own ideas!

Sample Responses to Discuss: Kadu

rope length is 4 yard

$\frac{3}{4} \times \pi 4^2 = 12\pi = 37.7\ yd^2$

a bit too small



Sample Responses to Discuss: Shifa

The shaded area of the grass is $\frac{3}{4}$ of a circle of radius r.

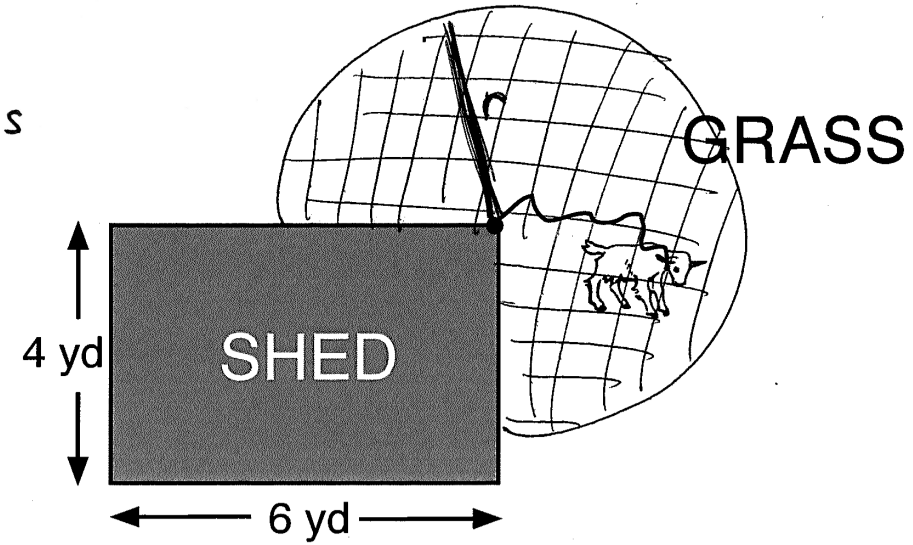
So $\frac{3}{4}\pi r^2 = 50$

$r^2 = \frac{50 \times 4}{3\pi}$

$r^2 = 21.2$

$r = \sqrt{21.2}$

$r = 4.6$ yards



Sample Responses to Discuss: Alice

length of rope = 6 yards

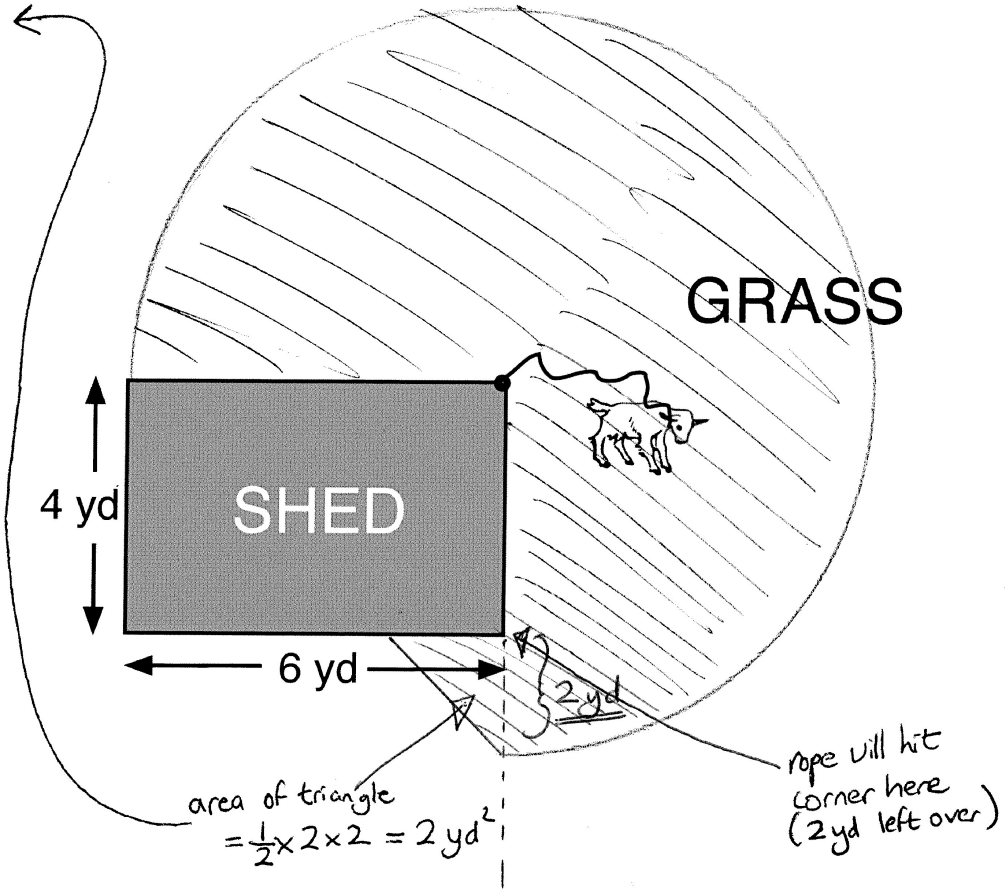
$\frac{3}{4}$ circle $= \frac{3}{4}\pi r^2 = \frac{3}{4}\pi 36$

$= 84.8$

triangle = 2

total area = $86.8\ yd^2$

too big!!!!



Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org