PROBLEM SOLVING

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Modeling: *Hot and Cold*

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Modeling: *Hot and Cold*

MATHEMATICAL GOALS

This lesson unit is intended to help students judge the accuracy of two different approximations to a particular linear relationship. Students will compare two linear functions as approximations to the relationship between Celsius and Fahrenheit temperature and consider under what circumstances each of the approximations may be reasonable.

COMMON CORE STATE STANDARDS

This lesson relates to **all** the *Standards for Mathematical Practices* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 1, 3, 4, 6, 7, and 8:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

7.EE: Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

INTRODUCTION

The lesson is structured in the following way:

- Before the lesson, students tackle the problem individually. You review their solutions and write questions to help students improve their work.
- At the beginning of the lesson, students respond to your questions. Students are then grouped into pairs or threes and work collaboratively to produce a better solution to the same task.
- There is a whole-class discussion to compare and evaluate different approaches.
- This is followed by a second collaborative activity in which students work in small groups to evaluate and comment on sample solutions, followed by a second whole-class discussion about the work.
- Finally, in a follow-up lesson, students review and evaluate their work on the problem.

MATERIALS REQUIRED

- Each student will need a copy of the assessment task *Hot and Cold*, a few sheets of paper, a calculator, the *How Did You Work?* questionnaire, and a mini-whiteboard, pen, and eraser.
- Each small group of students will need a large sheet of paper for making a poster, a felt-tipped pen, and copies of the *Sample Responses to Discuss*.
- You will need a supply of graph paper and rules available on request.

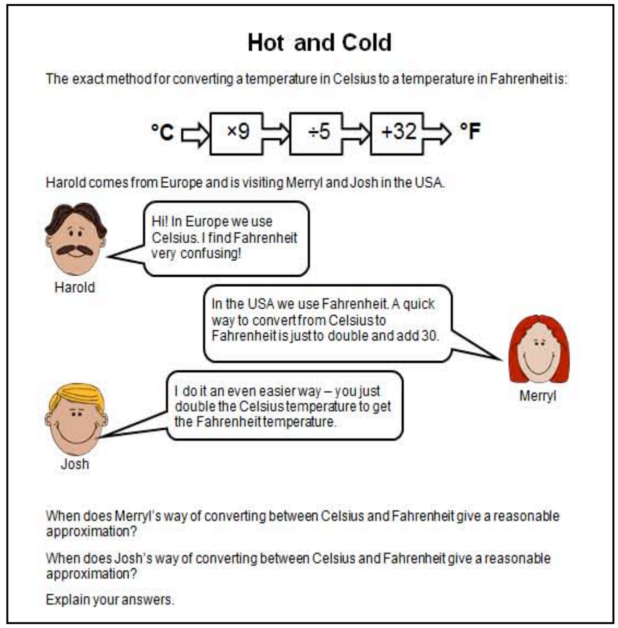
TIME NEEDED

Approximately 15 minutes before the lesson, a 90-minute lesson (or two 45-minute lessons), and 10 minutes in a follow-up lesson. Exact timings will depend on the needs of your students.

BEFORE THE LESSON

Assessment task: *Hot and Cold* (15 minutes)

Have students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.



Introduce the context and help the class to understand the problem:

What units are used to measure temperature?

Do you know of any other units for measuring temperature?

Students are likely to mention Fahrenheit, as this is common in the USA. Encourage students to refer to other units for measuring temperature that they may also know, for example, Celsius, which is common in Europe and Kelvin, which is used by scientists all over the world.

Do you know any benchmark temperatures for any of these scales? [For example, 100°C is the boiling point of water, 32°F is the temperature at which water freezes.]

Does anyone know how these different temperature scales are related to each other?

Why might we want to convert from one temperature scale to another? Do you know any way of converting from one to another?

Let students state any rules they know. Do not try to evaluate these or correct anything that is wrong at this stage. If students do not know anything about these temperature scales, that is fine.

Give each student a copy of the assessment task *Hot and Cold* and a sheet of paper to work on. Have graph paper and rules available for those who request them, but do not advertise this, in case it makes students think that they must use graph paper.

This task is about the Fahrenheit temperature scale and the Celsius temperature scale.
It describes how to convert from Celsius to Fahrenheit.

Merryl and Josh have come up with their own ways of converting from Celsius to Fahrenheit, which you are going to evaluate.

Read their statements and decide when Merryl's strategy gives a reasonable approximation to the exact value and when Josh's strategy seems to work. Explain your answer as fully as possible.

It is important that, as far as possible, students are allowed to answer the questions without assistance. If students are struggling to get started, then ask questions that help them understand what is required, but make sure you do not do the task for them.

Students who sit together often produce similar answers, then, when they come to compare their work, they have little to discuss. For this reason, we suggest that when students do the task individually, you ask them to move to different seats. At the beginning of the formative assessment lesson allow them to return to their usual seats. Experience has shown that this produces more profitable discussions.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem-solving approaches.

We suggest that you do not score students' work. Research suggests that this will be counterproductive, as it may encourage students to compare their scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students at the beginning of the lesson.

Common issues	Suggested questions and prompts
Fails to show any calculations For example: The student says that Josh's way is good because doubling is easy/when you're in a hurry. Or: The student says that neither way is useful to Harold because they begin with Celsius and go to Fahrenheit, rather than the other way around.	• How accurate are the answers you get doing it Josh's way? How could you check? • Have you tried Merryl's way? Which way gives a closer approximation to the exact value? How could you check? • Harold might want to convert Celsius temperatures that he is familiar with at home into Fahrenheit temperatures. Should he do it Merryl's or Josh's way? How could you find out?
Generalises from a few cases For example: The student tries out Merryl's and Josh's ways of converting on 0°C and 10°C and then states a conclusion.	• Will this always be the case? • What happens at other temperatures?
Forms a conclusion that does not depend on the range of temperatures considered For example: The student tries a few low values and says that Merryl's way is more accurate.	• Do your results follow a pattern? How could you check whether this pattern continues? • How many temperatures would be useful to test to be confident about your conclusion?
Considers only a limited temperature range For example: The student says that reasonable temperatures in the USA range from -10°C to 30°C.	• Can you think of a situation where Harold might require a temperature outside of this range?
Does not state a conclusion For example: The student only performs some calculations.	• Would you use Josh's or Merryl's way of converting or neither? Why?
Rejects both ways For example: The student says that both ways are inaccurate because they do not give the exact value.	• Are there any temperatures for which Merryl's or Josh's way do give the exact value? How could you check? • Are both ways of converting *equally* inaccurate or is one less bad than the other? How would you decide? • How near to the exact value would you consider to be a 'reasonable approximation'? Is this always the same? • Can you think of a situation when an approximation that is, for example, 2 degrees away from the exact value might be considered as reasonable and when it might not?

SUGGESTED LESSON OUTLINE

Individual review of the assessment task (10 minutes)

Return students' initial attempts at the assessment task and give each student a sheet of paper to work on. Begin the lesson by briefly reintroducing the problem.

Recall what we were looking at in a previous lesson. What was the task about?

If you did not add questions to individual pieces of work, write your list of questions on the board and ask students to select questions appropriate to their own work.

I looked at your work and I have some questions I would like you to think about.

On your own, carefully read through the questions I have written. I would like you to use the questions to help you to think about ways of improving your own work.

Use your sheet of paper to make a note of anything you think will help to improve your work. You will be explaining what you did initially and how it could be improved, to another student, later in the lesson when you work together to produce a joint solution.

Collaborative small-group work (25 minutes)

Organize the class into groups of two or three students. Give each group a large piece of paper and a felt-tipped pen. Have graph paper and rules available on request, but do not advertise this, as you do not want every student to think that they must draw a graph.

You are now going to work together to try to improve your initial attempts at the task.

Deciding on a Strategy

Use Slide P-1 of the projector resource to explain how students are to plan their joint method before they implement their agreed strategy:

Planning a Joint Method

1. Take turns to explain your method and how your work could be improved.
2. Listen carefully to each other. Ask questions if you don't understand.
3. When everyone in the group has explained their method, plan a joint method that is better than your separate ideas.
4. Make sure that everyone in the group can explain the reasons for your chosen method.
5. Write an outline of your method on your large sheet of paper.

At this stage you do not need to suggest strategies to students or worry if they do not think of things that you want them to do – this can come later. Do, however, encourage students to be specific about what they suggest, for example, if they say 'try it for some values', you could ask questions like:

What values? Why those values? What will you do with the answers you obtain? How will you decide whether the values are close enough? How will you use your answers to draw a conclusion?

Emphasize the need for students to be able to explain why they have chosen a specific method as well as being able to describe the strategy they intend to use. They might find it helpful to write their explanations on their poster along with their agreed method outline.

Implementing the Strategy

Once students have completed their plan on their large sheet of paper they need to turn it over so that they can use the other side to write their joint solution clearly in the form of a poster:

You are now going to implement your agreed strategy for solving the Hot and Cold task.

Turn your large sheet of paper over and use the other side to produce a joint solution.

While students work in small groups you have two tasks: to note different student approaches to the task and to support student problem solving.

Note different student approaches

Listen and watch students carefully. Note different approaches to the task and what assumptions students make. Do students work systematically? How do they organize their work? Are they concerned about the context, such as the purposes for which Harold might need to think about temperature? What do students do if they get stuck? Do they check their answers? In particular, note any common mistakes. You can then use this information to focus a whole-class discussion towards the end of the lesson.

Support student problem solving

Try not to make suggestions that move students towards a particular approach to the task. Instead, ask questions that help students clarify their thinking. In particular, focus on the strategies they are using, rather than the solution. Encourage students to justify their work.

If the whole-class is struggling on the same issue, you could write one or two relevant questions on the board and hold a brief whole-class discussion. You may want to use the questions in the *Common issues* table to support your own questioning.You could also give any struggling students one of the *Sample Responses to Discuss*.

You might, for example, suggest that they begin with Merryl's method and try some specific temperatures. You could ask:

Have you tried out Merryl's method on some specific values? Which temperatures did you try? How close was her method to the correct answer? Do you think that is close enough?

Whole-class discussion (10 minutes)

The purpose of this activity is to evaluate and compare different approaches.

I want us to share all the different ways you've interpreted the task.

You may have noticed some interesting ways of working or some incorrect methods. If so, you may want to focus the discussion on these. Equally, if you have noticed different groups using similar strategies, you may want to compare conclusions.

What method did you use? Was this helpful? Why / Why not?

What was the most challenging part of this task? Why?

What did you do to get 'unstuck'?

What did you learn when working together on the problem?

Encourage students to justify their work. For example:

Which temperatures did you test using Merryl's / Josh's way of converting? How did you decide?

How did you know when you had tried enough temperatures?

How did you use these calculations to determine when Merryl's / Josh's way of converting provides a reasonable approximation?

Extending the lesson over two days

If you are taking two days to complete the unit then you may want to end the first lesson here. At the start of the second day, briefly remind students of their previous work before moving on to the collaborative analysis of sample responses.

Collaborative analysis of *Sample Responses to Discuss* (30 minutes)

Give each student a mini-whiteboard, pen, and eraser and distribute copies of the *Sample Responses to Discuss* to each group of students. These give students an opportunity to evaluate a variety of possible approaches to the task, without providing a complete solution strategy. (Lara considers only Merryl's approximation and Matthew considers only Josh's.)

There may not be time, and it is not essential, for all groups to look at all three sample responses. If this is the case, be selective about what you hand out. For example, groups that have successfully completed the task using one method will benefit from looking at a different approach. Other groups that have struggled with a particular approach may benefit from seeing a sample student version of the same strategy.

Alternatively, if students are working in groups of three, it might be helpful to allocate each student with one of the three sample responses and then get students to re-group based on which sample response they have been allocated. They can then discuss that particular piece of work in detail in these larger groups before returning to their original groups of three to report back on what they have discussed.

> *In your groups you are now going to look at some student work on the task.*
> *Notice in which ways this work is similar to yours and in which ways it is different.*
>
> *There are some questions for you to answer as you look at the work.*
> *You may want to add notes to the work to make it easier to follow.*

Slide P-2 of the projector resource, *Evaluating Sample Student Responses*, describes how students should work together:

Evaluating Sample Student Responses

1. Take turns to work through a student's solution. Write your answers on your mini-whiteboard.
2. Explain your answers to the rest of the group.
3. Listen carefully to explanations. Ask questions if you don't understand.
4. When everyone is satisfied with the explanations, write the answers below the student's solution or on a separate piece of paper. **Make sure the student who writes the answers is not the student who explained them.**

Encourage students to focus on evaluating the math contained in the sample student work, not on its superficial appearance.

Emphasize the need for one student to write the responses to the questions whilst another student explains his or her thinking and then switching roles to promote student engagement with the work.

During the small group work, support the students as in the first collaborative activity. Also, check to see which of the explanations students find more difficult to understand.

Note similarities and differences between the sample approaches and those the students took in the collaborative group work.

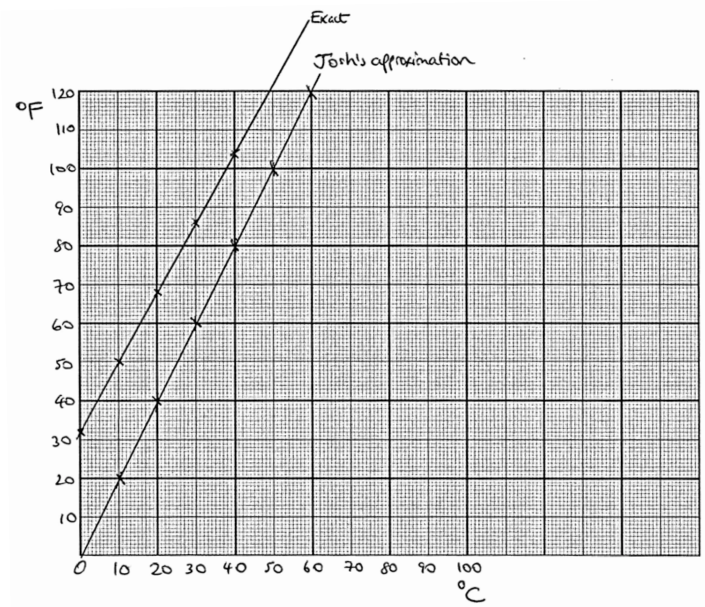
Lara calculates Merryl's approximation for Celsius temperatures going up in 10°C intervals from 0°C to 50°C. She finds the difference each time between Merryl's approximation and the exact Fahrenheit temperature. She seems to be finding the *absolute* difference, as she obtains an error of 2 for both 0°C and 20°C, even though in the first case Merryl's approximation is too low and in the second case it is too high.

Celsius temperature	Fahrenheit temperature	Merryl's approximation	How far out Merryl is
0	32	30	2
10	50	50	0
20	68	70	2
30	86	90	4
40	104	110	6
50	122	130	8

Merryl is not very far out, so her approximation is good.

Lara concludes that the errors are all small, but whether or not these numbers are small *enough* depends on the purpose for which the approximation is being used. A difference of 8°C may sound small, but could be significant in making a decision about what clothes you are going to wear, for instance. Lara does not consider temperatures outside her range or prove that the errors at the in-between temperatures lie in between the errors that she has calculated. She does not consider Josh's approximation at all.

Matthew takes a graphical rather than numerical approach. He plots values for Josh's approximation for Celsius temperatures going up in 10°C intervals from 0°C to 60°C. He also plots the exact Fahrenheit temperatures going up in 10°C intervals from 0°C to 50°C. He draws straight lines through these points, but it is not clear whether he knows that the in-between values will lie on these lines.



Josh's value is always too small.

Matthew concludes that Josh's approximation is always an *under*-approximation. However, the two lines appear to be converging, suggesting that they will cross at some higher temperature and from that point on Josh's value will always be too *high*. It may be that Matthew does not think that it is relevant how the approximation behaves at high temperatures, but he does not comment on this. He does not consider Merryl's approximation at all.

Ian considers both approximations and takes an algebraic approach. He defines his variable C explicitly and F, M, and J implicitly in terms of C, writing F, M and J as equations. Ian takes the difference between M and F as Merryl's error and the difference between J and F as Josh's error and writes these as equations.

Correct: $F = 1.8C + 32$

Merryl: $M = 2C + 30$

Josh: $J = 2C$

Where C = temp in Celsius

Merryl's error $= M - F = 0.2C - 2$

Josh's error $= J - F = 0.2C - 32$

$$0.2C - 32 < 0.2C - 2$$

So Josh's error is smaller than Merryl's error.

He concludes that Josh's error is smaller than Merryl's error. With error, we are normally concerned with the *absolute* difference – i.e.

we want to know which error is *closer to zero*. Taking this into account, it turns out that Merryl's approximation is closer to the true value for all temperatures up to 85°C and Josh's approximation is closer for all temperatures above 85°C (see solutions). At 85°C, both approximations are out by the same amount – Merryl's is 15°F above the exact value (185°F) and Josh's is 15°F below the exact value. However, the fact that one approximation is closer than another does not necessarily mean that it is close *enough* to be useful for any particular practical purpose!

Whole-class discussion: comparing different approaches (15 minutes)

Hold a whole-class discussion to consider the different approaches used in the sample responses. Focus the discussion on parts of the task students found difficult. Ask the students to compare the different solution methods.

Which approach did you like best? Why?

What were the advantages/disadvantages of that approach?

In what ways were the sample student responses incomplete?

Were any of the conclusions flawed in any way? Why?

Which approach did you find most difficult to understand? Why?

Students might comment on the way in which Matthew's graphical approach, for instance, makes the 'gap' between an approximation and the exact value very visible, but perhaps makes it harder to see individual values precisely. Plotting some points and ruling a line through them is not rigorous unless we know that the in-between values lie on that line.

On the other hand, Ian's algebraic approach enables us to identify the error explicitly as a function of the Celsius temperature, but absolute values are needed to avoid his false conclusion. Lara's numerical approach is a sensible way to begin but can be tedious if lots of values are considered and does not give us certainty for any values other than the ones we happen to try.

Perhaps combining Matthew's and Ian's approaches (i.e. drawing a graph for the error functions) would be a useful compromise.

Follow-up lesson: individual review (10 minutes)

Give out the *How Did You Work?* questionnaire and ask students to complete this, which should help them to review their progress.

If you have time, you may also want to ask your students to use what they have learned to attempt the task again. In this case, give each student a blank copy of the assessment task *Hot and Cold*.

Some teachers give this task as homework.

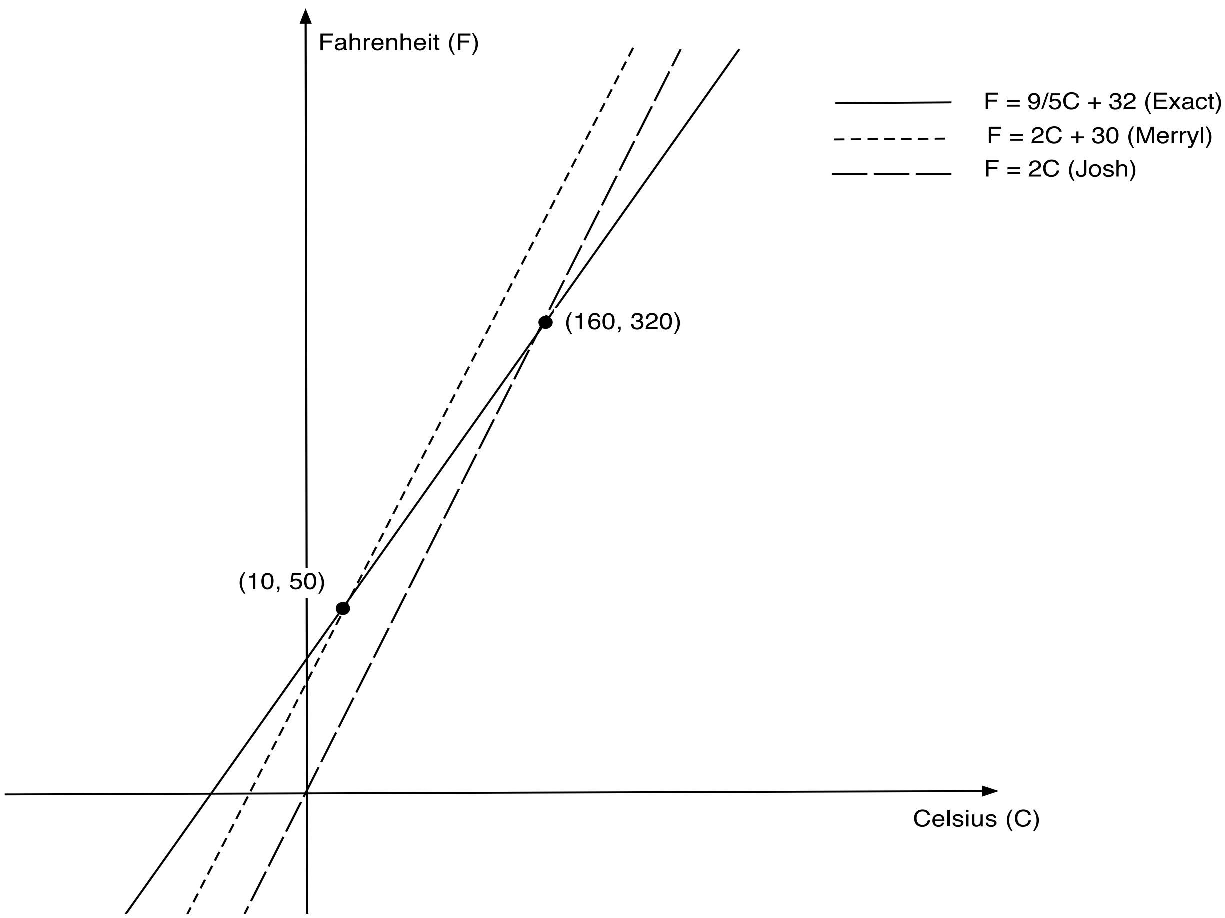
SOLUTIONS

There are a number of ways in which students could solve this problem precisely. Below is a possible solution strategy:

Writing the exact conversion from Celsius to Fahrenheit and Merryl's and Josh's approximations algebraically gives:

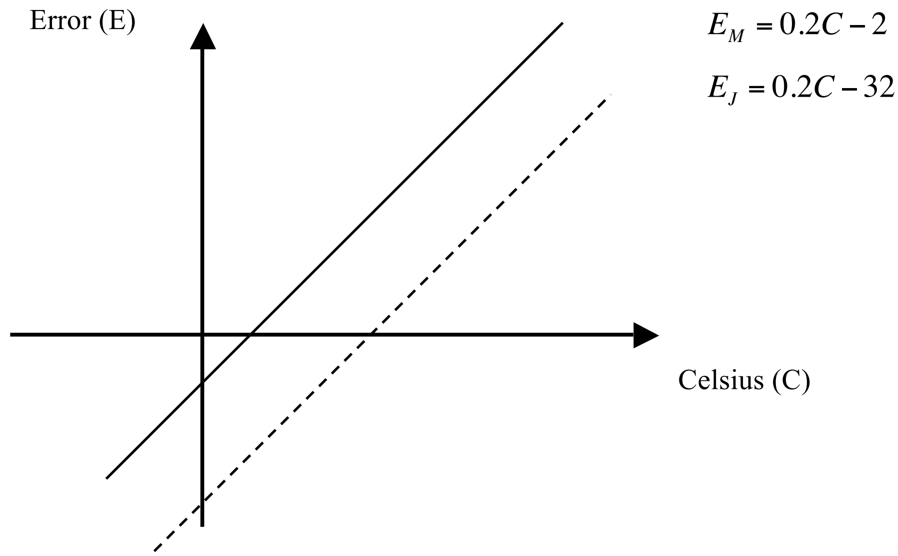
Exact conversion	$F = \frac{9}{5}C + 32$
Merryl	$F = 2C + 30$
Josh	$F = 2C$

Plotting these on a graph enables us to see the behavior of these three functions:

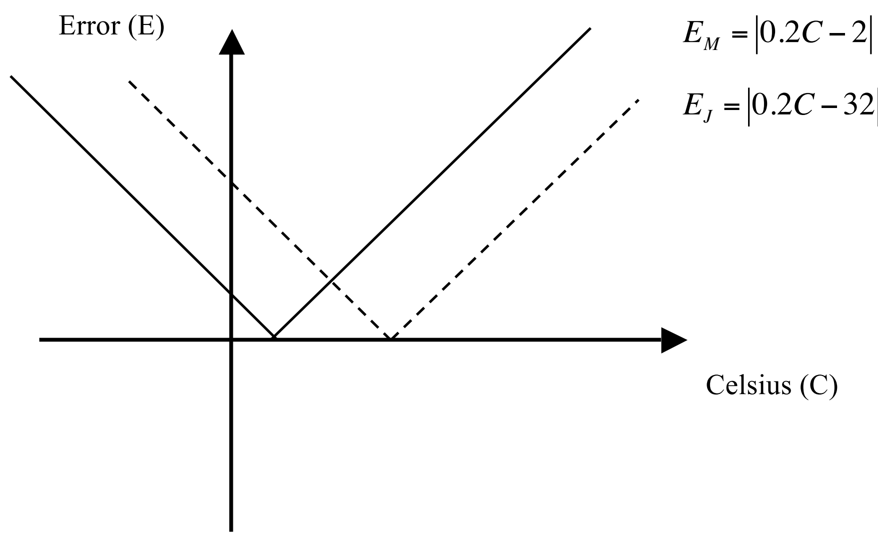


The function given by Merryl's approximation crosses the exact function at the point (10, 50). This means that at a temperature of 10°C Merryl's approximation gives the exact temperature of 50°F. Similarly, the function given by Josh's approximation crosses the exact function at the point (160, 320). This means that at a temperature of 160°C Merryl's approximation gives the exact temperature of 320°F. Approximations for temperatures near to these respective values will therefore be reasonably accurate suggesting that for lower temperatures, such as those encountered when thinking about the weather, Merryls' approximation may be better, whilst for higher temperatures, such as oven temperatures, for example, Josh's approximation may be more appropriate (and easier to carry out). The symmetry of this graph also shows that both approximations are at the same distance from the true value halfway between the points (10, 50) and (160, 320); that is at (85, 185).

Determining the points at which the approximations are too far away from the exact value to be considered 'close enough' for a particular practical application is also helpful and to do this we can find functions for the errors explicitly (like Ian did). Calling Merryl's error E_M and Josh's error E_J gives $E_M = 0.2C – 2$ and $E_J = 0.2C – 32$. These are linear functions, which can be graphed:



Since we are concerned with the magnitude of the error (the distance from zero), we need to be careful when interpreting these graphs. One way to do this is to reflect the negative portions of these graphs above the horizontal C axis, to obtain:



The point at which these two functions cross is the point at which Merryl's and Josh's approximation are the same amount away from the exact temperature.

Equating $E_M = 0.2C – 2$ with this reflected section of E_J $(32 – 0.2C)$ gives:
$0.2C – 2 = 32 – 0.2C$
$0.4C = 34$
$C = 85°$.

Substituting back into one of the functions, when C = 85° the error is 15°F with Merryl's approximation giving an overestimate of 200°F and Josh's approximation giving an underestimate of 170°F to the true value of 185°F.

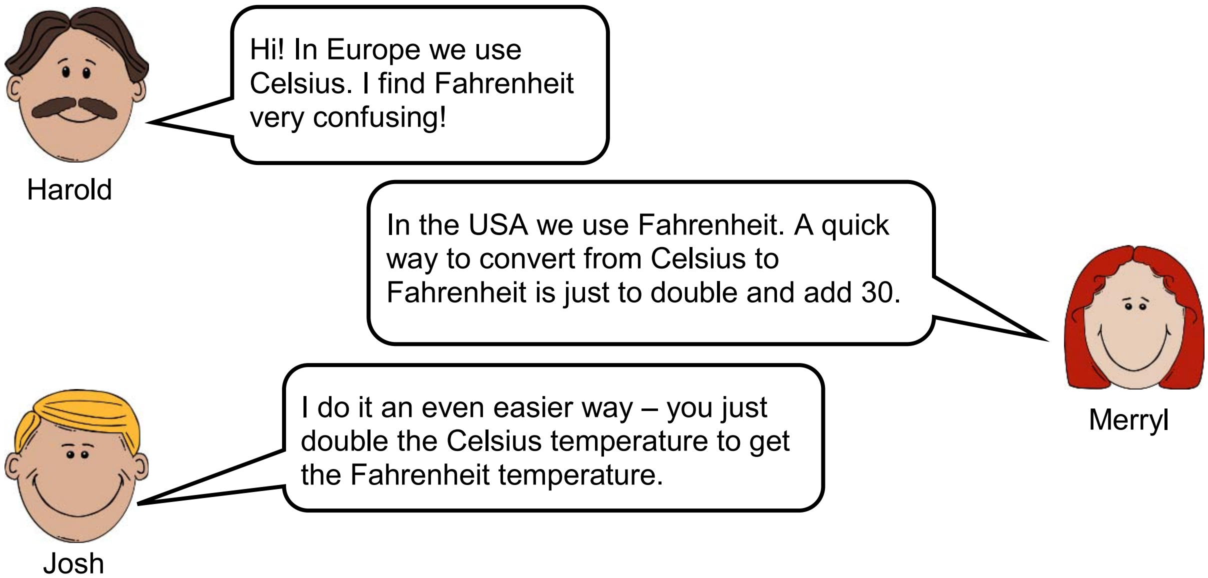
To the left of this intersection at 85°C, E_M is smaller than E_J, meaning that Merryl's approximation is closer to the true value; to the right of this intersection, E_J is smaller, meaning that Josh's approximation is closer to the true value. In fact the range for which Merryl's approximation lies within 15° of the true value is $-65° < C < 85°$ and for Josh's approximation the range is $85° < C < 235°$.

Hot and Cold

The exact method for converting a temperature in Celsius to a temperature in Fahrenheit is:

°C → ×9 → ÷5 → +32 → °F

Harold comes from Europe and is visiting Merryl and Josh in the USA.



When does Merryl's way of converting between Celsius and Fahrenheit give a reasonable approximation?

When does Josh's way of converting between Celsius and Fahrenheit give a reasonable approximation?

Explain your answers.

Sample Responses to Discuss: Lara

Lara has begun by looking at Merryl's approximation:

Celsius temperature	Fahrenheit temperature	Merryl's approximation	How far out Merryl is
0	32	30	2
10	50	50	0
20	68	70	2
30	86	90	4
40	104	110	6
50	122	130	8

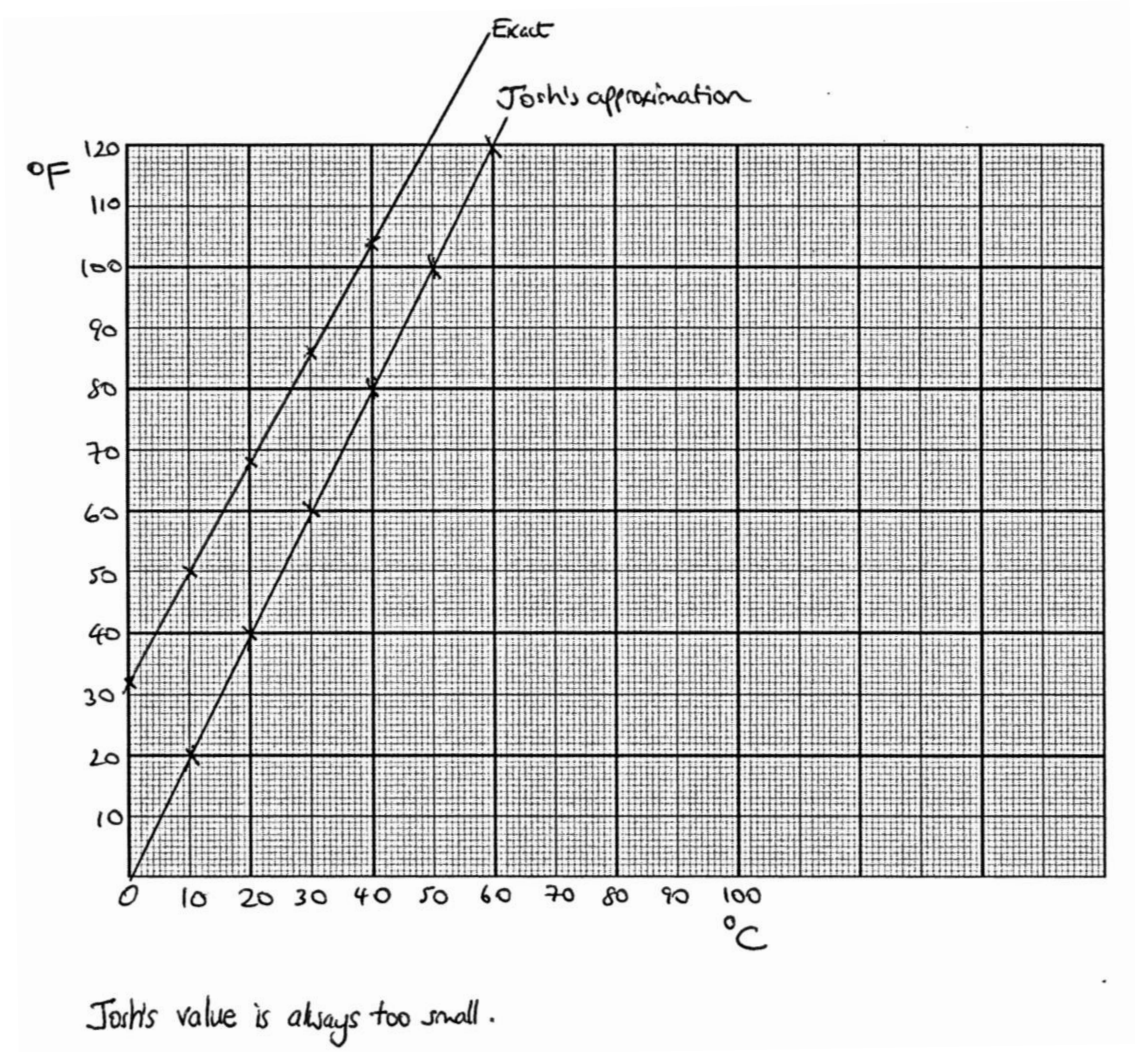
Merryl is not very far out, so her approximation is good.

1. Try to explain what Lara has done.

2. Do you agree with her conclusion? Why / Why not?

Sample Responses to Discuss: Matthew

Matthew has begun by looking at Josh's approximation:



1. Try to explain what Matthew has done.

2. Do you agree with his conclusion? Why / Why not?

Sample Responses to Discuss: Ian

Correct: $F = 1.8C + 32$

Merryl: $M = 2C + 30$

Josh: $J = 2C$

Where C = temp in Celsius

Merryl's error $= M - F = 0.2C - 2$

Josh's error $= J - F = 0.2C - 32$

$$0.2C - 32 < 0.2C - 2$$

So Josh's error is smaller than Merryl's error.

1. Try to explain what Ian has done.

2. Do you agree with his conclusion? Why / Why not?

How Did You Work?

Mark the boxes, circle an option and complete the sentences that apply to your work.

Yes/ No

1. Our group work was better than my own work

   This is because ....

2. Our solution is similar to one of the sample responses ☐ **OR** Our solution is different from **all** of the sample responses ☐

   Our solution is similar to .... *(add name of sample response)*

   I prefer **our solution / the sample response solution** (*circle*)

   This is because ....

   Our solution is different from all of the sample responses

   because ....

3. I made some assumptions ☐

   My assumptions were: ....

4. What advice would you give a student new to this task to help them with difficulties?

Planning a Joint Method

1. Take turns to explain your method and how your work could be improved.
2. Listen carefully to each other. Ask questions if you don't understand.
3. When everyone in the group has explained their method, plan a joint method that is better than your separate ideas.
4. Make sure that everyone in the group can explain the reasons for your chosen method.
5. Write an outline of your method on your large sheet of paper.

Evaluating Sample Student Responses

1. Take turns to work through a student's solution. Write your answers on your mini-whiteboard.

2. Explain your answers to the rest of the group.

3. Listen carefully to explanations. Ask questions if you don't understand.

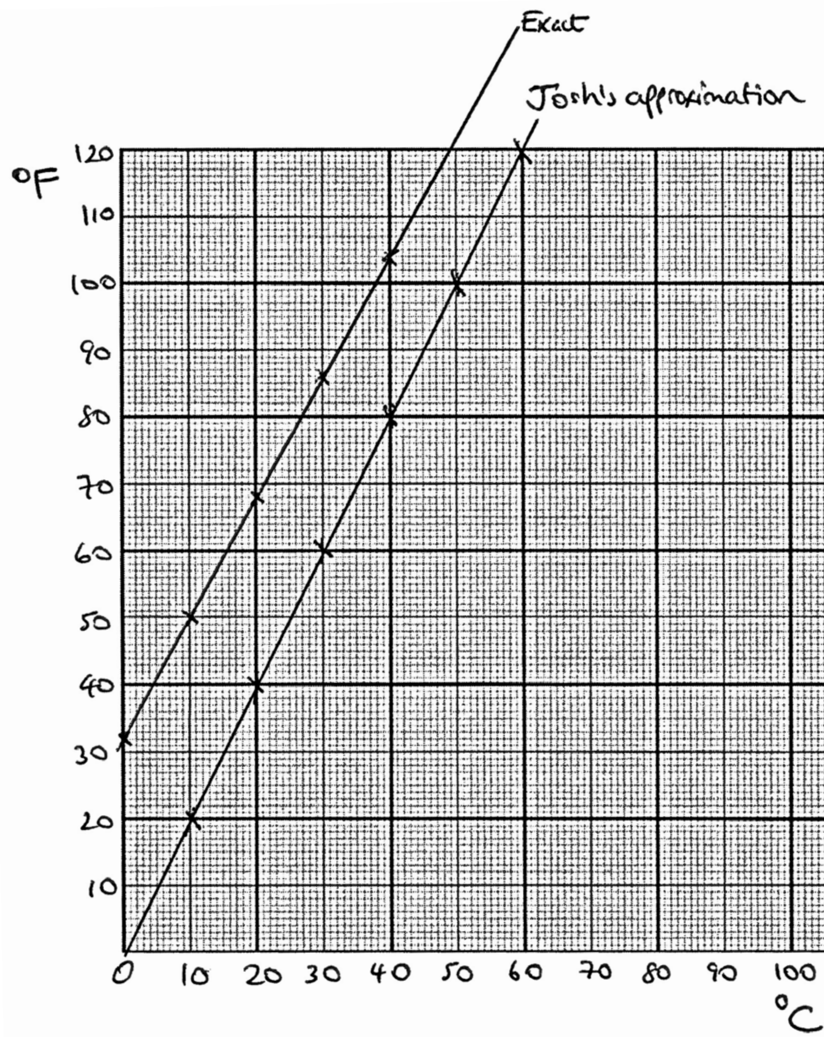
4. When everyone is satisfied with the explanations, write the answers below the student's solution or on a separate piece of paper. **Make sure the student who writes the answers is not the student who explained them.**

Sample Responses to Discuss: Lara

Celsius temperature	Fahrenheit temperature	Meryl's approximation	How far out Meryl is
0	32	30	2
10	50	50	0
20	68	70	2
30	86	90	4
40	104	110	6
50	122	130	8

Meryl is not very far out, so her approximation is good.

Sample Responses to Discuss: Matthew



Josh's value is always too small.

Sample Responses to Discuss: Ian

Correct: $F = 1.8C + 32$

Merryl: $M = 2C + 30$

Josh: $J = 2C$

where C = temp in Celsius

Merryl's error $= M - F = 0.2C - 2$

Josh's error $= J - F = 0.2C - 32$

$$0.2C - 32 < 0.2C - 2$$

So Josh's error is smaller than Merryl's error.

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
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with
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The full collection of Mathematics Assessment Project materials is available from
http://map.mathshell.org