PROBLEM SOLVING

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Representing:
Road Race

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Representing: *Road Race*

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to model a real-world situation and compare and critique sample models. In particular this lesson aims to identify and help students who have difficulty recognizing and using proportional relationships.

COMMON CORE STATE STANDARDS

This lesson relates to **all** the *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 2, 3, and 4:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

7.RP: Analyze proportional relationships and use them to solve real-world and mathematical problems.

INTRODUCTION

The lesson unit is structured in the following way:

- Before the lesson, students attempt the *A Race* task individually. You then assess their responses and formulate questions that will prompt students to review their work.
- At the start of the lesson, students think individually about their responses to the questions set.
- Next, students work in pairs or threes to combine their thinking and work together to produce a collaborative solution to the task, in the form of a poster.
- Working in the same small groups, students evaluate, comment on, and complete some sample responses.
- In a whole-class discussion students compare and evaluate the methods they have seen and used.
- Finally, in a follow-up lesson, students reflect on their work and what they have learned.

MATERIALS REQUIRED

- Each student will need a copy of the *A Race* task, some paper to work on, a mini-whiteboard, pen, and eraser, and a copy of the *How Did You Work?* questionnaire.
- Each small group of students will need a large sheet of paper, copies of the *Sample Responses to Discuss* and one or two post-it notes. Spare copies of the task may also be needed.
- Calculators and graph paper should be made available upon request.
- There is a projector resource to support whole-class discussions.

TIME NEEDED

20 minutes before the lesson, a 100-minute lesson (or two 50-minute lessons), and 15 minutes in a follow-up lesson. Exact timings will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *A Race* (20 minutes)

Have students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.

Give each student a copy of the *A Race* task. Check that students understand the context:

Has anyone seen/participated in a road race?

How many miles did competitors have to run? Did they run several laps of a route?

What are the advantages of a multi-lap race rather than just one long route? [More interesting for the spectators, easier to organize etc.]

Suppose a race was 10 miles, but the route was only 2 miles, how many laps would competitors run? [Five laps.]

What does it mean when one runner laps another runner? [One runner is so far ahead of another runner that they overtake and are one whole lap ahead.]

If students are still unsure about what it means for one runner to lap another, you may like to ask a couple of students to demonstrate the process by acting it out for the rest of the class.

In this task there is a race between two people - Amy and Rebecca.

Read the questions carefully and answer them as fully as you can.

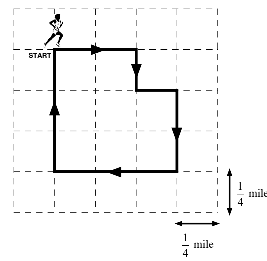
It is important that, as far as possible, students work on the task without your assistance. If students are struggling to get started then ask questions that help them understand what is required, but make sure you do not do the task for them.

Students who sit together often produce similar answers and then when they come to compare their work, they have little to discuss. For this reason, we suggest that when students do the task individually, you ask them to move to different seats. Then at the beginning of the formative assessment lesson, allow them to return to their usual seats. Experience has shown that this produces more profitable discussions.

When all students have made a reasonable attempt at the task, tell them that they will have time to revisit and revise their solutions later.

A Race

Amy and Rebecca are running in a road race. The map, drawn to scale, shows the route of the race:



The race consists of four laps of the route and Amy and Rebecca run clockwise along the route at a constant speed.

It takes Amy 8 minutes to run a mile.
Rebecca takes 12 minutes to run a mile.

1. Mark on the map where Rebecca '**R**' and Amy '**A**' will be one hour into the race. Explain how you know.

....

....

....

2. Will one runner 'lap' the other runner at some point in the race? If so, where? Label this place '**X**' and explain your reasoning. If not, explain how you know.

....

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem solving approaches.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores and will distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this lesson unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students at the beginning of the lesson.

Common issues	Suggested questions and prompts
Struggles to get started	• What can you tell me about the race? • In your head, use your own words to describe what the question is asking.
Confuses laps and miles For example: The student labels Amy as being at the starting position and Rebecca as being half way round the route, as Amy will have done 5 laps and Rebecca will have done $7\frac{1}{2}$ laps (Q1).	• How far is a lap in miles? • How long does it take Amy/Rebecca to complete a lap?
Interprets the positions of Amy and Rebecca incorrectly For example: The student interprets the positions of Amy and Rebecca after an hour on their diagram as being that Rebecca is ahead of Amy. Or: The student compares 8 minutes to 12 minutes and concludes that Rebecca will be in front because the bigger figure means she is faster.	• Suppose Amy and Rebecca raced just one block. Who would get to the finish first? • How many laps will Amy/Rebecca have run after an hour?
Does not provide reasoning For example: The student states that Amy will lap Rebecca because she runs faster, but does not attempt to justify this mathematically (Q2).	• Explain in words what you did to arrive at your solution and/or show your work. • What mathematics do you still need to work out that will help others understand your thinking?
Makes incorrect assumptions For example: The student assumes that Rebecca and Amy won't lap because if they are both running at a constant speed, they will always be the same distance apart (Q2).	• How far apart are Amy and Rebecca when they have completed one lap? What about when they have completed two laps?
Uses an unsystematic or inefficient method For example: The student works out some times and distances for each runner, but does not organise them (Q2). Or: The student performs many calculations without noticing any patterns (Q2).	• You have calculated some distances and times. Do you notice anything from your figures? How could that help you explain this problem? • How can you organize your figures to make it easier to understand what is happening and what you are noticing about them? • Look at the times and distances you have already calculated. Do you notice any patterns in those calculations?
Successfully completes the task The student needs an extension activity.	• If you were organizing the race, describe how you could adapt it so that the two runners would have a close finish.

SUGGESTED LESSON OUTLINE

Reviewing individual solutions to the problem (10 minutes)

Give each student a mini-whiteboard, a pen, and an eraser and return their work on the *A Race* task.

If you have not added questions to individual pieces of student work, either give each student a printed version of your list of questions with the questions that relate to their work highlighted, or write your list of questions on the board so that students can select questions from the board that are appropriate to their own work.

Recall what we were working on previously. What was the task about?

I have looked at your work and I have some questions I would like you to think about.

On your own, carefully read through the questions I have written.
I would like you to use the questions to help you think about ways of improving your work.

Use your mini-whiteboards to make a note of anything you think will help to improve your work.
You will be sharing these notes with a partner later on.

If mini-whiteboards are not available or so that there is a permanent record of their thinking, students may want to use the back of their assessment task or a clean piece of paper to jot down their ideas about ways to improve their work.

This is an opportunity for students to review their own work before working collaboratively on producing a group solution.

While students are reviewing their work, it may be appropriate to ask individual students questions that help them to clarify their thinking. However, the purpose of the activity is not to address misconceptions; there will be opportunities for students to deal with these collaboratively, later in the lesson.

Collaborative small-group work: making posters (30 minutes)

Organize the class into groups of two or three students.

Today you are going to work together on the task to produce a joint solution that is better than your individual work.

Before students make another attempt at the task, they need to discuss what they have learned from reviewing their individual solutions.

Show and explain to students Slide P-2 of the projector resource:

Sharing Individual Solutions

1. Take turns to share your individual work with your partner(s).
2. Share the notes you made on how you might improve your work.
3. Listen carefully to each other, asking questions if you don't understand.
4. Notice any similarities or differences between the methods described.

Once students have had chance to discuss their work, hand out to each group a sheet of poster paper.

Display Slide P-3 of the projector resource and explain to students how to work collaboratively to produce a joint solution:

Joint Solution: Making Posters

1. In your group, agree on the best method for completing the problem.
2. Produce a poster that shows a joint solution to the *A Race* task, that is better than your individual work.
3. State on your poster any assumptions you have made.
4. Give clear reasons for your choice of method.

Emphasize to students that, providing they justify their decisions, they can improve one of their individual methods, combine methods or use a completely different method. It is important that their joint solution builds upon the thinking and knowledge gained from producing individual solutions and is not merely a copy of their original work. It should be a joint effort and an improved version of everyone's individual solutions.

While students are working on a joint solution you have two tasks: to note different student approaches to the task and to support student problem solving.

Note different student approaches

In particular, notice whether students' original methods are the same or different. If they are different, how do they decide which method to use for their joint solution? What are their reasons for the choice of method? Are students aware of any assumptions they have made? Do they justify these assumptions? How do they organize their work? Do they notice any patterns in their calculations? Are they concerned whether their answer makes sense?

You may also want to notice particular approaches such as non-standard ones, ones that include a common misconception, ones that are particularly efficient or inefficient.

How does your method help you to explain your answer to question two?

Can you justify your solution?

How might you explain your solution without referring to the diagram of the track?

You may decide to draw on some of these approaches when conducting the whole-class discussion later in the lesson.

Support student problem solving

If students are struggling to produce a joint solution to the task, try not to make suggestions that move them towards a particular approach. Instead, ask questions to help students clarify their thinking, encouraging them to identify the strengths and weaknesses of the methods employed in their individual solutions. Can any of these methods be improved to produce a group solution that is better than the original individual response? Can they think of any other approaches to try?

What have you done that you both [all] agree on? Why have you chosen this method?

What else do you need to find out?

What do you now know that you didn't know before?

If several students in the class are struggling, you might want to write a relevant question on the board or hold a brief whole-class discussion. You may also want to use some of the questions in the *Common issues* table to support your own questioning. In particular, students who have made good progress on the task should be encouraged to extend the task to think about ways of adapting the race to ensure a close finish for the two runners.

Sharing posters (10 minutes)

Once students have finished their posters, give each group one or two post-it notes. Ask students to share their work by visiting another group. This gives all students the opportunity to actively engage with another group's reasoning, as well as voicing their own thinking.

Display Slide P-4 and explain how students are to share their work and the purpose of the activity:

Sharing Posters

1. One person from each group get up and visit a different group.
2. If you are staying with your poster, explain your work to the visitor, giving reasons for your choice of method.
3. If you are the visitor, look carefully at the work, asking clarifying questions to help you to understand the method used.
4. Discuss whether or not the method described on the poster is similar to the visitor's method.
5. The visitor is to write on a post-it note, suggestions on how the work could be improved.

Extending the lesson over two days

If you are taking two days to complete the unit then you may want to end the first lesson here. At the start of the second day, briefly remind students of the work they have already done before moving on to the collaborative analysis of sample responses.

Collaborative analysis of *Sample Responses to Discuss* (30 minutes)

Once students have had sufficient time to share and discuss their joint solutions, distribute to each group, copies of the *Sample Responses to Discuss*. If you do not have sufficient time to give students all three pieces of work, then select two that are appropriate to your class. Ensure all have access to the same responses. We have found from trials that if students work on different solutions, the whole-class discussion can be too challenging!

Show and explain to students Slide P-5 of the projector resource:

Sample Responses to Discuss

1. Read each piece of sample student work carefully.
2. Try to understand what they have done. You may want to add annotations to the work to make it easier to follow.
3. Take turns explaining your thinking to your partner.
4. Listen carefully and ask clarifying questions.
5. When your group has reached its conclusions, write your answers to the questions underneath the work.
6. Finally, compare the sample responses.
   - What are the strengths and weaknesses of each?
   - Which do you prefer?
   - Justify your answers.

Emphasize to students that the purpose of the activity is not to check the accuracy of the math as none of the methods contain errors, but to understand and evaluate a variety of possible approaches to the task and to notice any differences and/or similarities with their own work.

The task encourages students to be flexible in their approach and recognize relationships among different points of view and so deepen their mathematical understandings.

Students should thoughtfully answer the questions below each piece of sample student work and be encouraged to think carefully about ways in which the work could be improved.

Sally's work is incomplete. She uses the fact that when Amy laps Rebecca, their times are the same and they are 3 miles apart.

She correctly uses proportional reasoning to figure out the distances.

Her method is fairly efficient and clear.

Sally now needs to describe the position where the two participants are on the track.

Rebecca 12 mins 1 mile
24 mins 2 miles

Amy 8 mins 1 mile
24 mins 3 miles

For Amy to lap Rebecca she needs to be 3 miles in front

I will use guess and check

Time	Rebecca	Amy	
24	2	3	1 mile in front x
48	4	6	2 miles in front x
96	8	12	4 miles in front x

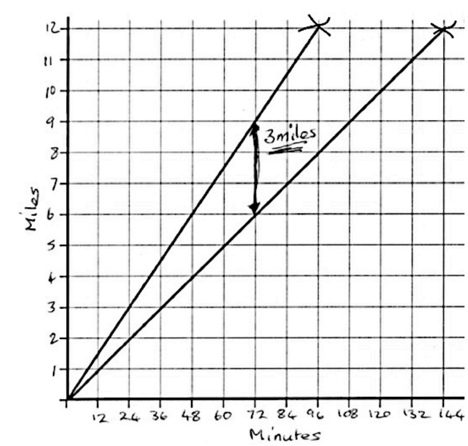
Diane's work is incomplete. She has used a graph to represent the race.

She correctly figures out the time it takes Amy and Rebecca to complete the race.

She correctly figures out the time Amy laps Rebecca (when they are 3 miles apart).

Diane has not shown her working and she has not labeled the lines.

Diane now needs to clearly describe the position of Amy and Rebecca at 72 minutes.



George uses blocks instead of miles. He correctly states Amy must be 12 blocks in front of Rebecca when she laps her.

George correctly uses proportional reasoning to figure out when they will be 12 blocks apart. He does not justify his method.

George now needs to clearly describe the position of Amy and Rebecca at 72 minutes.

When Amy laps Rebecca they will be 12 blocks apart

1 block - 2 minutes (Amy)
3 minutes (Rebecca)

6 minutes
Amy : run 3 blocks
Rebecca : run 2 blocks) 1 block apart

A difference of 12 blocks will take 12×6
= 72 minutes

Whole-class discussion: comparing different approaches (20 minutes)

Hold a whole-class discussion to consider the different approaches used within the sample responses. Look at each response in turn and ask students to comment on their strengths and weaknesses.

If you have time, you may also want to include in the discussion some of your own students' responses to the problem. It may be helpful to display Slides P-6 to P-8 during this discussion.

What did each student do?

Once you have discussed each piece of work, ask students to compare and evaluate the different methods.

Which piece of work did you find easiest/most difficult to understand? Why was that?

Which method do you prefer? Please explain.

Suppose each method was clearly explained. Which method would you now prefer? Please explain.

What method would you prefer to use for a 1-lap race/20-lap race? Why?

How were the sample student responses similar/different to what you did?

Did analyzing the responses enable anyone to see ways in which they could improve their own work? Please explain.

Follow-up lesson: individual reflection (15 minutes)

Once students have had a chance to discuss the sample responses as a whole-class, distribute the questionnaire *How Did You Work?*

Ask students to spend a couple of minutes, individually, answering the questions.

Think carefully about your work this lesson and the different methods you have seen and used.

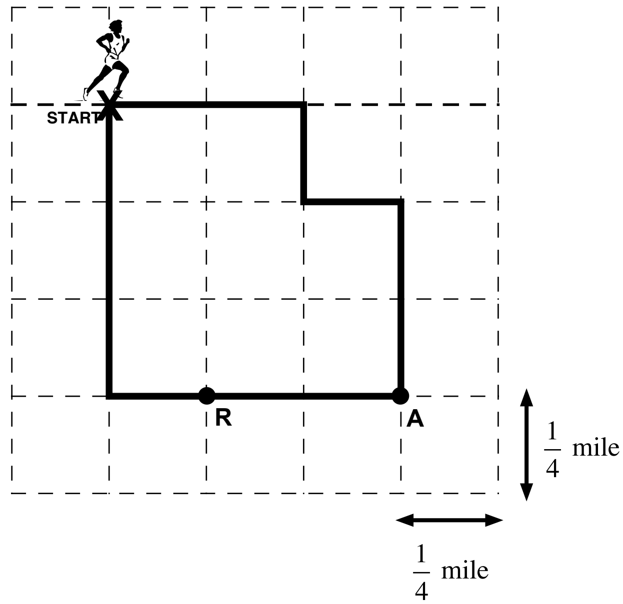
On your own, answer the review questions as carefully as you can.

Encourage students to give detailed answers to the questionnaire, based on the mathematical detail of the approaches seen and used. In particular, students should include in their responses a comparison of the strengths and weaknesses of the different approaches contained within this lesson unit.

Some teachers give this as homework.

SOLUTIONS

1. An hour into the race Amy (**A**) and Rebecca (**R**) will be at the positions shown below:



One lap is 3 miles (the course is equivalent to a square of side length 3 blocks).

Amy will be ahead of Rebecca by 2½ miles. She will have run 7½ miles (2 laps and 6 blocks) and Rebecca will have run 5 miles (1 lap and 8 blocks). So after 1 hour Rebecca will be 2 blocks further round the course than Amy.

2. Amy will lap Rebecca when Amy has run 3 laps. At this time, Rebecca will just be completing her second lap.

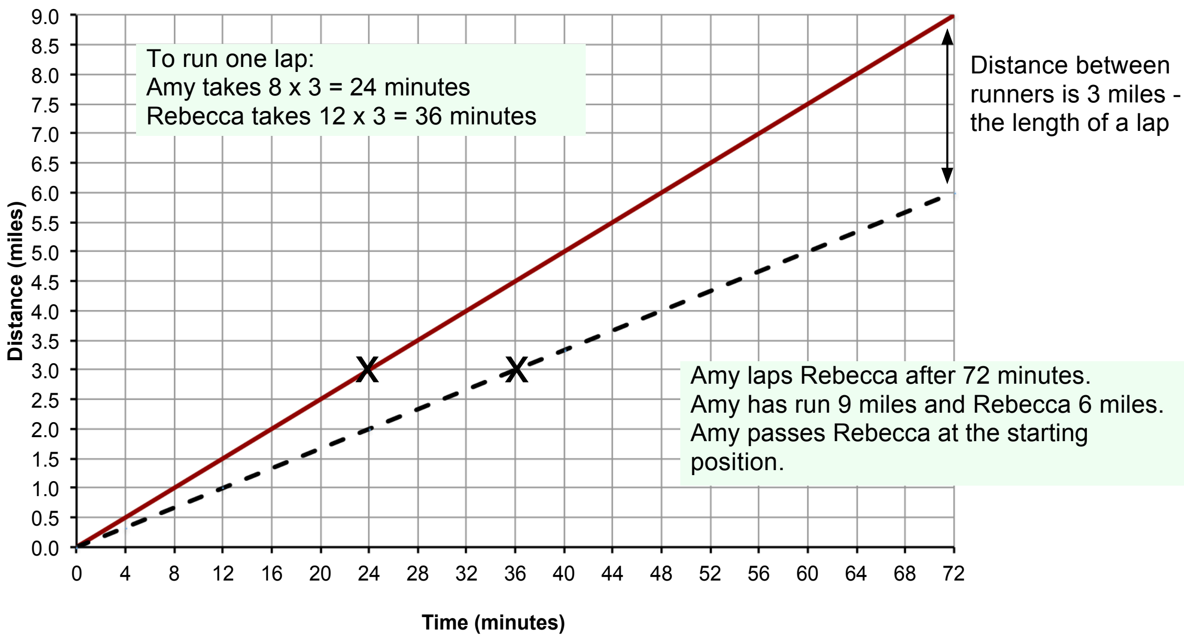
Below are four possible methods students may use. Instead of miles, some students may prefer to work in 'block'.

Method 1: A table

Time (minutes)	Distance (miles): Amy	Distance (miles): Rebecca	Difference in distances (miles)
0	0	0	0
12	1.5	1	0.5
24	3	2	1
36	4.5	3	1.5
72	**9**	**6**	**3**

When Amy laps Rebecca, she has completed 3 laps and Rebecca has completed 2 laps. The point X should be at the start of the course.

Method 2: Graphical approach



Method 3: Relative velocity

This is a challenging method, and it is not likely that students will use it. However, you may draw students' attention to it when they are discussing the graphical method.

After 24 minutes, Amy has run 3 miles and Rebecca has run 2 miles.
Every 24 minutes the distance between Amy and Rebecca increases by 1 mile.
They are 3 miles apart after 24 × 3 = **72 minutes**.

Method 4: Algebra.

After t minutes Amy has run $\frac{t}{8}$ miles.

After t minutes, Rebecca has run $\frac{t}{12}$ miles.

They pass when

$$\frac{t}{8} = \frac{t}{12} + 3$$
$$\Rightarrow \frac{t}{24} = 3$$
$$\Rightarrow t = 72$$

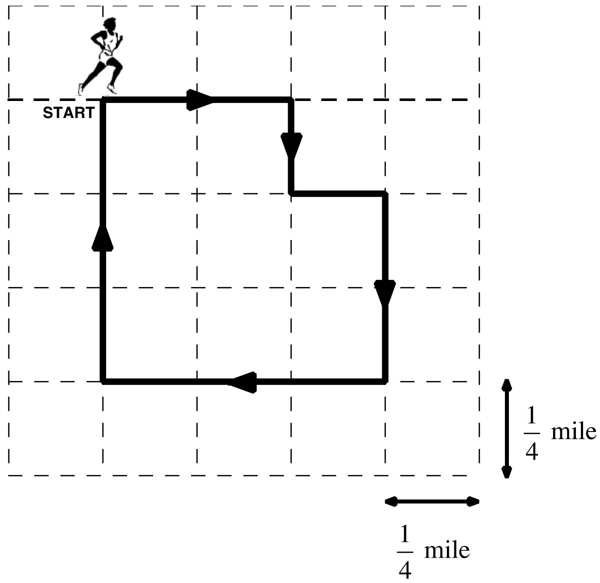
That is, after 72 minutes.

Method 5: Calculating speed

Since Amy runs 2.5 mph faster than Rebecca she will get 1 lap (3 miles) ahead of Rebecca after 3/2.5 = 1.2 hours. It is important to check that this happens before the race has finished, otherwise it is not a valid solution! So we calculate that Amy will have run 7.5 x 1.2 = 9 miles and Rebecca will have run 5 x 1.2 = 6 miles and we see that Amy is indeed 1 lap (3 miles) ahead of Rebecca and neither has exceeded the total 12 miles length of the race course. Since both runners have travelled a multiple of 3 miles it follows that Amy will lap Rebecca at the start point on the course.

A Race

Amy and Rebecca are running in a road race. The map, drawn to scale, shows the route of the race:



The race consists of four laps of the route and Amy and Rebecca run clockwise along the route at a constant speed.

It takes Amy 8 minutes to run a mile.
Rebecca takes 12 minutes to run a mile.

1. Mark on the map where Rebecca '**R**' and Amy '**A**' will be one hour into the race.
   Explain how you know.

2. Will one runner 'lap' the other runner at some point in the race?
   If so, where? Label this place '**X**' and explain your reasoning.
   If not, explain how you know.

Sample Responses to Discuss: Sally

Rebecca 12 mins 1 mile
24 mins 2 miles

Amy 8 mins 1 mile
24 mins 3 miles

For Amy to lap Rebecca she needs to be 3 miles in front

I will use guess and check

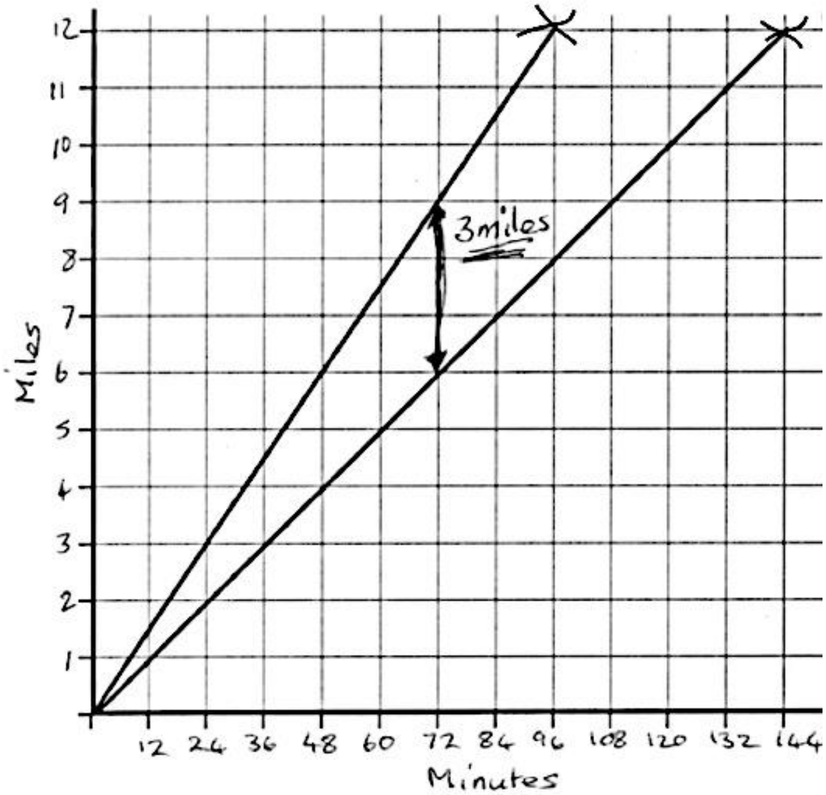
Time	Rebecca	Amy
24	2	3
48	4	6
96	8	12

1 mile in front x
2 miles in front x
4 miles in front x

Summarize Sally's method as clearly as you can.

Complete Sally's work.

Sample Responses to Discuss: Diane



Clarify Diane's work, by adding more detail to the graph.
Explain how Diane worked out the position of the two lines.

Complete Diane's work.

Sample Responses to Discuss: George

When Amy laps Rebecca they will be 12 blocks apart

1 block - 2 minutes (Amy)
3 minutes (Rebecca)

6 minutes
Amy : run 3 blocks
Rebecca : run 2 blocks) 1 block apart

A difference of 12 blocks will take 12×6
= 72 minutes

Clearly explain George's reasoning, including a check that his answer is correct.

Complete George's work.

How Did You Work?

Mark the boxes, circle an option and complete the sentences that apply to your work.

1. Our method was similar to one of the sample responses ☐ **OR** Our method was different from **all** of the sample responses ☐

Our method was similar to ____________ *(add name of sample response)*

because ....

....

....

....

Our method was different from all of the sample responses

because ....

....

....

....

2. Of all the different methods I've seen, I prefer ____________________ (*add name of sample response*) method because: ....

....

....

3. His / Her method could be improved by ....

....

....

4. Our method could be improved by ....

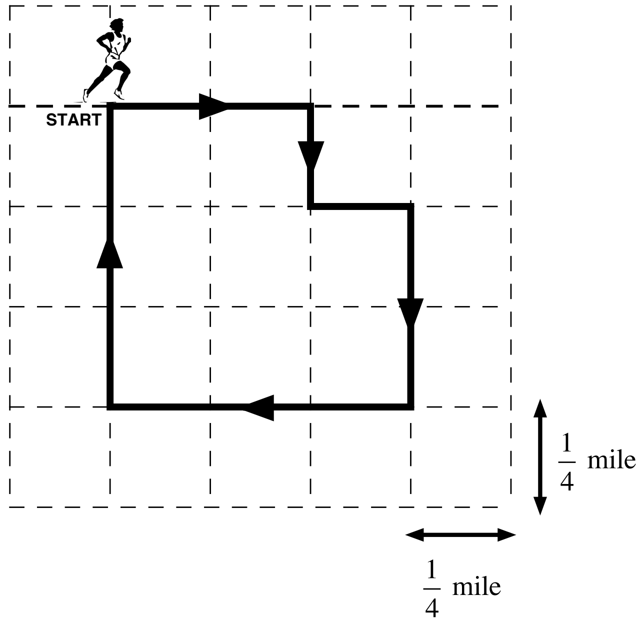
....

....

....

A Race

Amy and Rebecca are running in a road race.
The map, drawn to scale, shows the route of the race:



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Sharing Individual Solutions

1. Take turns to share your individual work with your partner(s).

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Joint Solution: Making Posters

1. In your group, agree on the best method for completing the problem.

2. Produce a poster that shows a joint solution to the *A Race* task, that is better than your individual work.

3. State on your poster any assumptions you have made.

4. Give clear reasons for your choice of method.

Sharing Posters

1. One person from each group get up and visit a different group.
2. If you are staying with your poster, explain your work to the visitor, giving reasons for your choice of method.
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5. When your group has reached its conclusions, write your answers to the questions underneath the work.
6. Finally, compare the sample responses.
    - What are the strengths and weaknesses of each?
    - Which do you prefer?
    - Justify your answers.

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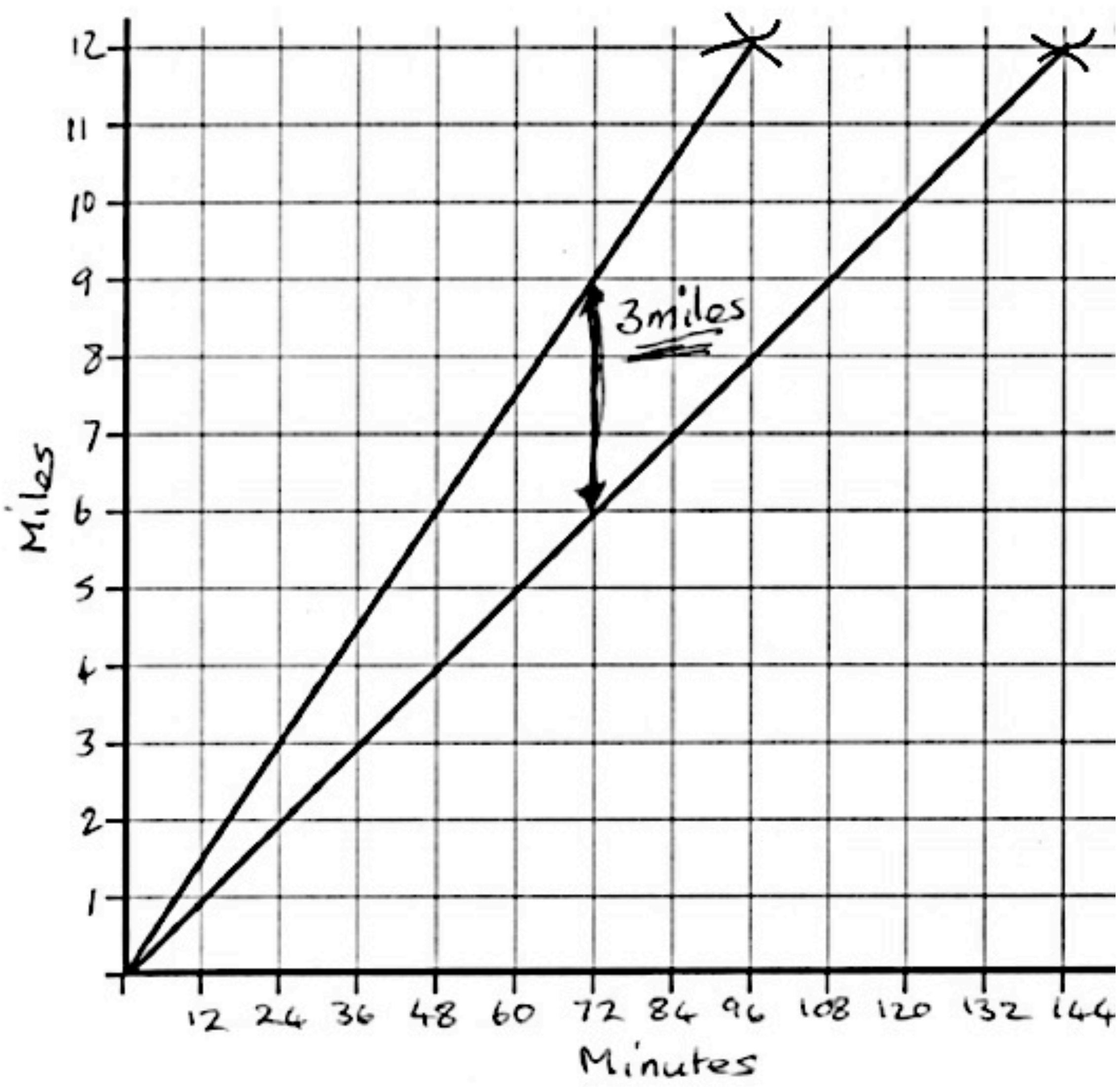
Sample Responses to Discuss: Diane



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Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
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with
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Hugh Burkhardt, Daniel Pead, and **Malcolm Swan** at the University of Nottingham

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The full collection of Mathematics Assessment Project materials is available from
http://map.mathshell.org