CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Using Proportional Reasoning

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Using Proportional Reasoning

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to reason proportionally when comparing the relationship between two quantities expressed as unit rates and/or part-to-part ratios. In particular, it will help you assess how well students are able to:

- Describe a ratio relationship between two quantities.
- Compare ratios expressed in different ways.
- Use proportional reasoning to solve a real-world problem.

COMMON CORE STATE STANDARDS

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

6.RP: Understand ratio concepts and use ratio reasoning to solve problems.

This lesson also relates to **all** the *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 1, 2, 3, 4, and 6:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

The lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task designed to reveal their current understanding and difficulties. You then review their solutions and create questions for students to consider, in order to improve their work.
- After a whole-class introduction, students work in groups, putting diagrams and descriptions of orange and soda mixtures into strength order. Students then compare their work with their peers.
- Next, in a whole-class discussion, students critique some sample work stating reasons why two mixtures would or wouldn't taste the same. Students then revise and correct any misplaced cards.
- After a final whole-class discussion, students work individually either on a new assessment task, or return to the original task and try to improve their responses.

MATERIALS REQUIRED

- Each student will need a mini-whiteboard, pen, and eraser, and a copy of *Mixing Drinks* and *Mixing Drinks (revisited)*.
- Each small group of students will need the cut-up *Card Set: Orange and Soda Mixtures* and *Card Set: Blank Cards*, a sheet of poster paper and a glue stick.
- You may wish to have some orange juice and soda for mixing/tasting but this is not essential.

TIME NEEDED

15 minutes before the lesson, a 100-minute lesson (or two 55-minute lessons), and 15 minutes in a follow-up lesson. Timings given are approximate and will depend on the needs of your class.

BEFORE THE LESSON

Assessment task: *Mixing Drinks* (15 minutes)

Have students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the lesson that follows.

Give each student a copy of *Mixing Drinks*. Introduce the task briefly, helping the class to understand the task:

> *This task is about making a fizzy orange drink by mixing different quantities of orange and soda.*
>
> *You are going to compare how orangey the drinks will taste, as well as working out the amount of orange and soda needed to make fizzy orange with a similar orangey taste.*

Mixing Drinks

When Sam and his friends get together, Sam makes a fizzy orange drink by mixing orange juice with soda.

On Friday, Sam makes 7 liters of fizzy orange by mixing 3 liters of orange juice with 4 liters of soda.

On Saturday, Sam makes 9 liters of fizzy orange by mixing 4 liters of orange juice with 5 liters of soda.

1. Does the fizzy orange on Saturday taste the same as or different to Friday's fizzy orange?

   If you think it tastes the same, explain how you can tell.
   If you think it tastes different, does it taste more or less orangey? Explain how you know.

2. On Sunday, Sam wants to make 5 liters of fizzy orange that tastes *slightly less orangey* than Friday's and Saturday's fizzy orange. For every liter of orange, how many liters of soda should be added to the mixture? Explain your reasoning.

It is important that, as far as possible, students answer the questions on the sheet without assistance. If students are struggling to get started, ask questions that help them understand what they are being asked to do, but do not do the task for them.

Students should not worry too much if they cannot understand or do everything, because there will be a lesson related to this, which should help them. Explain to students that by the end of the next lesson they should expect to answer questions such as these confidently; this is their goal.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem-solving approaches.

We suggest that you do not score students' work. Research suggests that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics. Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this unit.

We recommend that you:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions, and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these questions on the board when you return the work to the students in the follow-up lesson.

Common issues	Suggested questions and prompts
Reasons additively rather than multiplicatively For example: The student states that the fizzy orange tastes the same on Saturday as it did on Friday because one more liter of orange and one more liter of soda has been added and these just 'cancel each other out' (Q1). Or: The student states that the fizzy orange tastes the same on Saturday as it did on Friday because both mixtures contain one more liter of soda than orange (Q1).	• How could you use math to check that the addition of a liter of orange and a liter of soda has no effect on the taste? • What would happen to the taste if a liter of orange and a liter of soda were added to 1 liter of soda? • If 3 liters of fizzy orange was made in the same way, by mixing 1 liter of orange with 2 liters of soda, would this taste the same also?
Sole focus on orange as the 'active' ingredient For example: The student thinks that Saturday's fizzy orange will taste more orangey than Friday's, because it has more orange in it than Friday's has (Q1).	• How much soda is in Saturday's fizzy orange? How much soda is in Friday's fizzy orange? What do you notice? • Is how orangey the fizzy orange tastes determined by the number of liters of orange it contains?
Sole focus on soda as the diluting ingredient For example: The student thinks that Saturday's fizzy orange will taste less orangey than Friday's, because it has more soda in it than Friday's has (Q1).	• How much orange is in Saturday's fizzy orange? How much orange is in Friday's fizzy orange? What do you notice? • If 5 liters of fizzy orange were made by mixing 4 liters of soda with 1 liter of orange, would it also taste more orangey than Saturday's fizzy orange?
Provides an explanation based on one mixture only For example: The student states that Saturday's fizzy orange will taste less orangey than Friday's, because the mixture contains less orange in it than soda (Q1).	• Does Friday's fizzy orange contain more orange than soda or more soda than orange? • How can you compare the taste of Saturday's fizzy orange to the taste of Friday's fizzy orange?
Makes incorrect assumptions For example: The student thinks that on Sunday, Sam should mix 1 liter of orange with 4 liters of soda because 2 liters of orange with 3 liters of soda will taste the same as Friday's and Saturday's fizzy orange (Q2). Or: The student assumes that for every liter of orange two liters of soda are required (Q2).	• Will this fizzy orange mixture taste *slightly* less orangey than Friday's and Saturday's fizzy orange?
Provides little mathematical explanation	• Can you use math to explain your answer?
Completes the task correctly The student needs an extension task.	• Can you find a fizzy orange mixture that is more orangey than Friday's fizzy orange but less orangey than Saturday's fizzy orange?

SUGGESTED LESSON OUTLINE

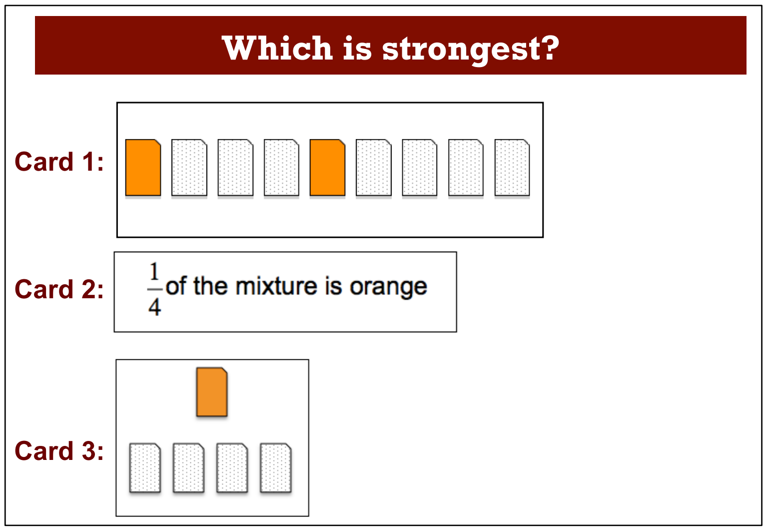
Whole-class introduction (10 minutes)

Give each student a mini-whiteboard, pen, and eraser. Remind the class of the assessment task they have already attempted.

Recall what we were working on previously. What was the task about?

In today's lesson we are going to consider different mixtures of orange and soda used to make fizzy orange and think about which ones taste more/less orangey.

Display Slide P-1 of the projector resource:



Each of these three cards describes a fizzy orange mixture.

The diagrams on cards 1 and 3 show the amount of orange and soda in the mix (where the shaded boxes represent the orange and the dotted boxes represent the soda) and card 2 gives a description of the fraction of the fizzy orange mixture that is orange.

Working on your own, on your mini-whiteboard, write the card numbers in order from least orangey to most orangey. [Card 3, Card 1, Card 2.]

Give students a few minutes to work on this before asking to see their whiteboards. If there are a range of responses within the class, collate them on the board and hold a whole-class discussion. Spend a few minutes discussing the strategies used to compare the three cards.

Explain to students that they are going to be working in groups on a similar activity putting cards in order of strength from least orangey to most orangey.

Individual think time, then collaborative work: *Orange and Soda Mixtures* (30 minutes)

Before students work collaboratively, it can be helpful to give students individual 'thinking time'. This allows everyone to have time to construct ideas to share and avoids the conversation being dominated by one student.

Organize students into groups of two or three. Give each group the cut-up *Card Set: Orange and Soda Mixtures*, a sheet of poster paper, and a glue stick.

On these cards there are descriptions of fizzy orange mixtures.
Some cards show the number of orange and soda juice boxes in the mixture, some contain a written description of the mixture and some show empty juice boxes which you will need to shade in (color orange juice boxes and draw dots for soda.)

Display Slide P-2 of the projector resource:

Individual think time

Your task is to work with your partner to put the cards in order of strength, from least orangey (on the left) to most orangey (on the right).

1. Look at the cards and think about ways you could carry out this task.

2. Write your ideas on your mini-whiteboards.

There is no need for students to order the cards during this individual activity.

When students have had sufficient time to think about the task:

First, take turns to explain to each other your ideas for how to carry out the task.

Ask questions if you do not understand your partner's explanation.

Take a few minutes to come up with a joint plan of action.

Display Slide P-3 of the projector resource and explain how students are to work together on the task:

Working together

1. Work together to put the cards in order of strength, taking turns with the work.
   a. Explain decisions to your partner.

2. If you think more than one card describes the same fizzy orange mixture, group them together.
   a. If a group of cards does not contain a juice box card, then shade in one of the Cards M - P.

3. When you both agree where each card should go and why, glue them onto your poster. On your poster, explain your decisions.

While students are working, you have two tasks: to notice their approaches to the task and to support student problem solving.

Make a note of student approaches to the task

Listen and watch students carefully. In particular, notice how students make a start on the task, where they get stuck, and how they overcome any difficulties.

Do they begin with what they think is the strongest or weakest mixture or do they just pick a random card? Do students compare orange to soda (e.g. for every orange there are 2 soda) or orange to mixture (e.g. ½ the mixture is orange). When they discover cards that are of equal strength, how do they justify this to one another? Do they use fractions, decimals, percentages, ratios or proportions? Do they switch between different descriptions? How do they go about shading cards M to P?

You can use this information to focus a whole-class discussion towards the end of the lesson.

Support student problem solving

As students work on the task support them in working together. Encourage them to take turns and if you notice that one partner is doing all the ordering or that they are not working collaboratively on the task, ask a student in the group to explain a card placed by someone else in the group.

Try not to make suggestions that push students towards a particular approach to the task. Instead, ask questions to help students clarify their thinking. The following questions and prompts may be helpful:

Which mixture do you think is the most orangey? Why?

How do you know that this mixture is more orangey than that one?

Why does this card come here?

Encourage students to write on the cards.

If several students in the class are struggling with the same issue, you could write one or two relevant questions on the board and hold a brief whole-class discussion. For example, if students are using 'additive' rather than 'multiplicative' reasoning; e.g. thinking that 3:5 (Card B) is the same as 4:6 (Card E) you could ask:

Why do you think that these will taste the same?

Can you think of another fizzy orange mixture that will also taste the same? How do you know?

Students who finish early with the cards in the right order could be given cut-up *Card Set: Blank Cards* and asked:

Can you invent a card that would go in between these two?

Can you invent a card that would go in the same place as this one?

What would you add to this mixture to make it taste like this mixture?

Sharing work (15 minutes)

Give students the opportunity to compare their work by visiting another group. It is likely that some groups will not have ordered all the cards but a comparison can still be made as to whether students consider a particular card to be more orangey or less orangey than another. It may be helpful for students to jot down on their mini-whiteboards their agreed order of the cards before they visit another group.

Show Slide P-4 and explain how students are to share their work:

Sharing work

1. One person from each group get up and visit a different group.
2. If you are staying with your poster, explain your card order to the visitor, justifying the placement of each card.
3. If you are the visitor, look carefully at the work and challenge any cards that you think are in the wrong place.
4. If you agree on the placement of the cards, compare your methods used when ordering.

Extending the lesson over two days

If you are taking two days to complete the unit then you may want to end the first lesson here. At the start of the second day, allow time for students to remind themselves of their work before moving on to discuss their ordering of the cards as a whole-class.

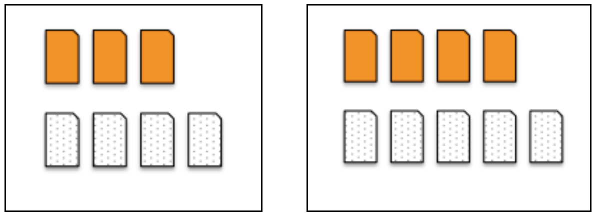
Whole-class discussion (25 minutes)

Now hold a brief whole-class discussion in which students discuss their ordering. Draw attention to significant differences between the ordering that particular groups have arrived at.

Were there any disagreements when you compared your work? Someone give me an example. What reasoning did you each give?

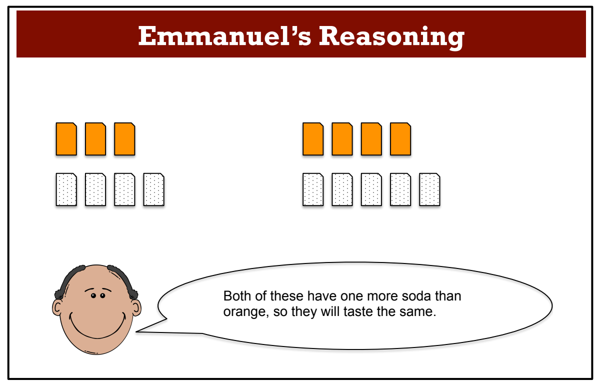
Was different math used to figure out the ordering? [E.g. orange to soda or orange to mixture]

Once you have had a chance to compare reasons given, spend some time exploring conflicting reasoning/conclusions when comparing the following two fizzy orange mixtures:



Display Slide P-5 of the projector resource to show Emmanuel's reasoning and ask:

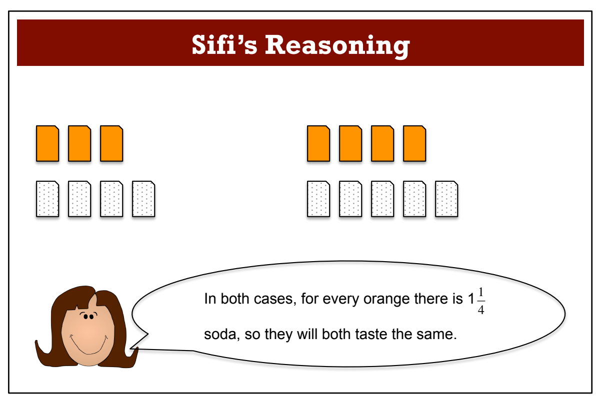
What do you think about Emmanuel's reasoning? Is he right or wrong? Why?



Students should be suspicious of this kind of 'linear' reasoning by now and if they are not you could explore what would happen if you continued the pattern to the left (2 orange and 3 sodas, 1 orange and 2 sodas etc.). Taking one more step to the left we would have no orange and 1 soda. There is still 'one more soda than orange' but everyone will agree that this will not taste orangey at all!

Now display Slide P-6 of the projector resource showing Sifi's reasoning and ask:

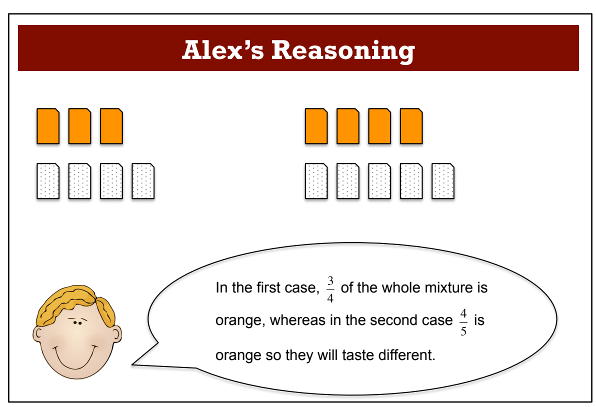
What do you think about Sifi's reasoning? Is she right or wrong? Why?



Sifi's method is better than Emmanuel's because she is thinking proportionately, but she has made an error; $1\frac{1}{4}$ is correct for the right-hand mixture, for the number of soda juice boxes per orange juice box, but the left-hand mixture is $1\frac{1}{3}$.

Now use Slide P-7 of the projector resource to display Alex's reasoning and ask:

What do you think about Alex's reasoning? Is he right or wrong? Why?



Alex has come to the correct conclusion about the mixtures not tasting the same but his method contains an error. The left-hand mixture is $\frac{3}{7}$ orange not $\frac{3}{4}$ orange (it is in the ratio of 3:4 (orange:soda)) and the right-hand mixture is $\frac{4}{9}$ orange not $\frac{4}{5}$ (ratio 4:5). Since $\frac{3}{7}$ is less than $\frac{4}{9}$, the right-hand mixture will be *slightly* more orangey (but it may be hard to tell this small difference in practice!)

Finally, you might want to ask:

Did you use any of these methods? Which ones?

Did you use any other methods? What were they?

What do you think now about all of these methods?

Poster review (10 minutes)

Students now have an opportunity to reconsider the ordering of their cards:

Now that you have had a chance to compare and discuss your work and we have looked at what Emmanuel, Sifi and Alex have said, you might like to have another look at your poster and decide in your groups whether you are still happy with where you have placed the cards.
If you think a card is in the wrong place, draw an arrow on your poster to where you think it should go.

While this is happening, encourage students to voice their reasoning for the movement of a card.

Whole-class discussion (10 minutes)

You may want to finish with a brief whole-class discussion in which students discuss their ordering and talk more generally about what they have gained from the lesson.

Did you change your ordering after we talked together about it? Why / Why not?

How confident are you with your ordering now?

What have you learnt today about how you get mixtures that taste the same or different?

Use your knowledge of the students' group work to call on a wide range of students for contributions.

Follow-up lesson: reviewing the assessment task (15 minutes)

Give students their responses to the original assessment task *Mixing Drinks* and a copy of the task *Mixing Drinks (revisited)*. If you have not added questions to individual pieces of work then write your list of questions on the board. Students then select from this list only those questions they think are appropriate to their own work.

Look at your original responses and the questions [on the board/written on your paper]. Answer these questions and revise your response.

On your mini-whiteboard make some notes on what you have learned during the lesson. Now have a go at the second sheet: Mixing Drinks (revisited). Can you use what you have learned to answer these questions?

If students struggled with the original assessment task, you may feel it more appropriate for them to revisit *Mixing Drinks* rather than attempting *Mixing Drinks (revisited)*. If this is the case give them another copy of the original assessment task instead.

If you are short of time you could give this task for homework.

SOLUTIONS

Assessment task: *Mixing Drinks*

1. The ratio of orange to soda on Friday is 3:4, which is not equal to the ratio of orange to soda on Saturday (4:5), so the fizzy orange mixtures will not taste the same. Friday's mixture is $\frac{3}{7}$ orange and Saturday's mixture is $\frac{4}{9}$ orange. Comparing these fractions to see which will taste the most orangey ($\frac{3}{7} = \frac{27}{63}$ compared with $\frac{4}{9} = \frac{28}{63}$) reveals that Saturday's fizzy orange mixture will taste more orangey. However, students may comment that even though Saturday's fizzy orange is stronger than Friday's, it is likely that you would not be able to taste any difference because the difference is only very slight.
2. If Sam mixes 2 liters of orange with 3 liters of soda, the mixture will be $\frac{2}{5}$ orange, which is *slightly* less orangey than Friday's and Saturday's mixture. This means that for every liter of orange, $1\frac{1}{2}$ liters of soda should be added to the mixture.

Assessment task: *Mixing Drinks (revisited)*

1. The completed table is as follows: (missing values are identified in **bold**)

Amount of Raspberry Juice (liters)	Amount of Apple Juice (liters)	Amount of Soda (liters)	Total Amount of *Fabulous Fruit Fizz* (liters)
1	2	3	6
0.5	1	**1.5**	**3**
2	**4**	**6**	12

2. a. $\frac{2}{5}$ of the drink is apple juice.

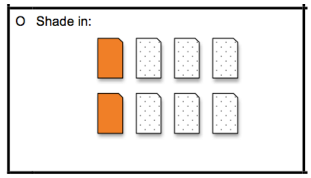
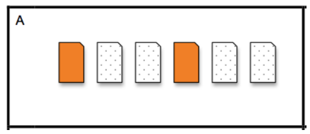
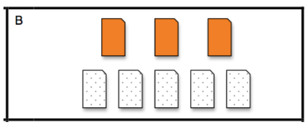
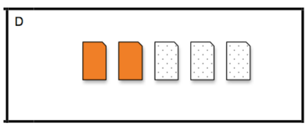
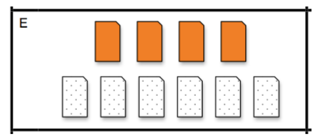
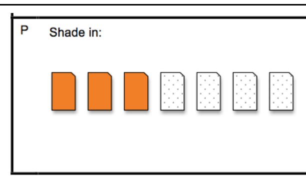
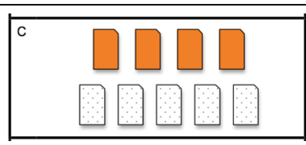
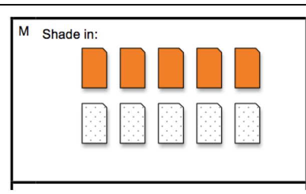
   b. $\frac{2}{5}$ of the drink is apple juice.

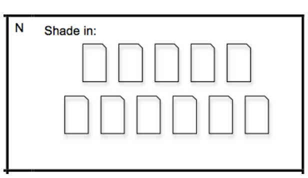
   c. $\frac{1}{3}$ of the drink is apple juice.

   Mixture c is the least appley drink. Qaylah should mix, for every liter of apple, 2 liters of soda.

Collaborative task:

The correct matching/ordering from least orangey to most orangey (with ratio of orange to soda also given) is as follows:

1:3	I One fourth of the mixture is orange		
1:2	J $\frac{2}{3}$ of the mixture is soda		G For every orange there are 2 sodas
3:5			
2:3		L For every soda there is $\frac{2}{3}$ orange	
3:4	K For every orange there is $1\frac{1}{3}$ soda		
4:5		H Orange : Soda = 4 : 5	
1:1	F Half of the mixture is orange		



Card N has been designed so that it cannot be shaded to be equivalent to any of the other cards. Students should shade the card with a number of orange/soda juice boxes of their choice (between 1 and 10) and then place it in the appropriate place based on how orangey the mixture is.

For example, they may choose to shade it in the ratio of 5 orange: 6 soda and place it between cards C and M.

Mixing Drinks

When Sam and his friends get together, Sam makes a fizzy orange drink by mixing orange juice with soda.

On Friday, Sam makes 7 liters of fizzy orange by mixing 3 liters of orange juice with 4 liters of soda.

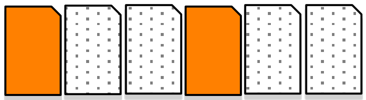
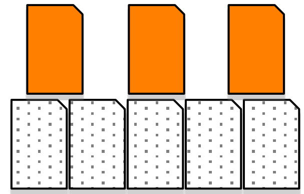
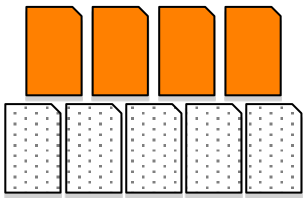
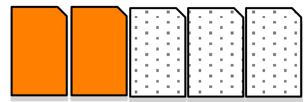
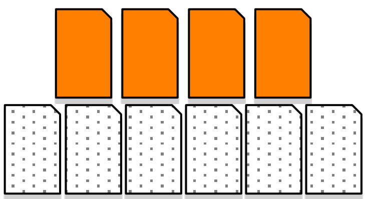
On Saturday, Sam makes 9 liters of fizzy orange by mixing 4 liters of orange juice with 5 liters of soda.

1. Does the fizzy orange on Saturday taste the same as Friday's fizzy orange, or different?

   If you think it tastes the same, explain how you can tell.
   If you think it tastes different, does it taste more or less orangey? Explain how you know.

2. On Sunday, Sam wants to make 5 liters of fizzy orange that tastes *slightly* less orangey than Friday's and Saturday's fizzy orange. For every liter of orange, how many liters of soda should be added to the mixture? Explain your reasoning.

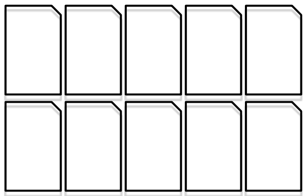
Card Set: Orange and Soda Mixtures

A 	B 
C 	D 
E 	F **Half of the mixture is orange**
G **For every orange there are 2 sodas**	H **Orange : Soda = 4 : 5**
I **One fourth of the mixture is orange**	J **$\frac{2}{3}$ of the mixture is soda**
K **For every orange there is $1\frac{1}{3}$ soda**	L **For every soda there is $\frac{2}{3}$ orange**

Card Set: Orange and Soda Mixtures (continued)

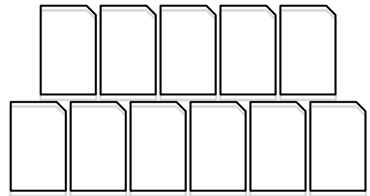
M

Shade in:

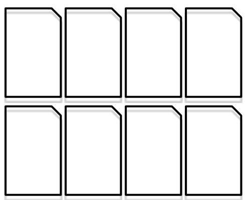


N

Shade in:

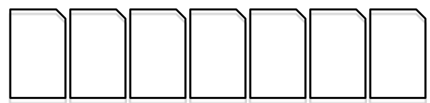


O Shade in:



P

Shade in:



Card Set: Blank Cards

Mixing Drinks (revisited)

To make 6 liters of *Fruit Fizz*,
mix 1 liter of raspberry juice, 2 liters of apple juice and 3 liters of soda

1. Complete the table below with the amounts of raspberry juice, apple juice and soda needed to make the different quantities of *Fruit Fizz*. The mixture must taste exactly the same each time.

Amount of Raspberry Juice (liters)	Amount of Apple Juice (liters)	Amount of Soda (liters)	Total Amount of *Fruit Fizz* (liters)
1	2	3	6
	1		
			12

2. Here are three ways to make apple fizz:

   a. For each liter of soda mix $\frac{2}{3}$ liters of apple juice.

   b. Mix apple and soda in the ratio 2 : 3.

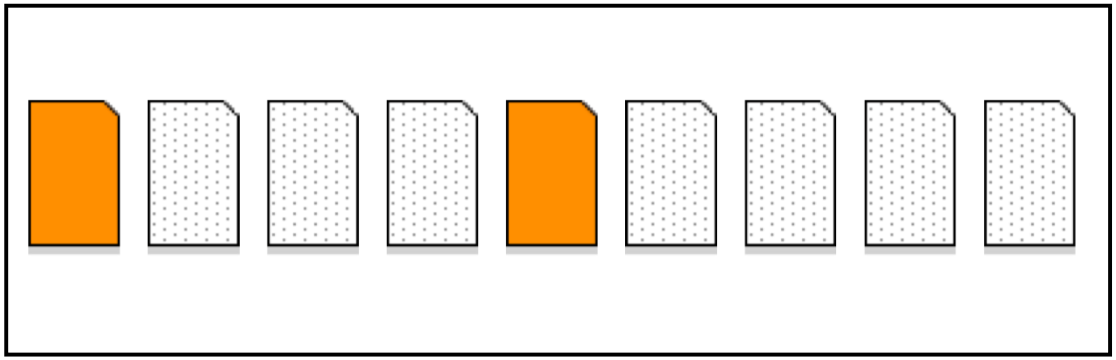
   c. $\frac{2}{3}$ of the mixture is soda, the rest is apple juice.

   Qaylah wants to mix the least appley drink. Which mixture should she choose?

   For every liter of apple, how many liters of soda should she add to the mixture?
   Explain your reasoning.

Which is strongest?

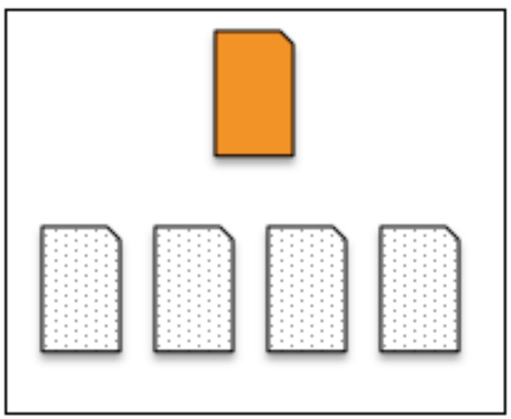
Card 1:



Card 2:

$\frac{1}{4}$ of the mixture is orange

Card 3:



Individual think time

Your task is to work with your partner to put the cards in order of strength, from least orangey (on the left) to most orangey (on the right).

1. Look at the cards and think about ways you could carry out this task.

2. Write your ideas on your mini-whiteboards.

Working together

1. Work together to put the cards in order of strength, taking turns with the work.
   a. Explain decisions to your partner.

2. If you think more than one card describes the same fizzy orange mixture, group them together.
   a. If a group of cards does not contain a juice box card, then shade in one of the Cards M - P.

3. When you both agree where each card should go and why, glue them onto your poster. On your poster, explain your decisions.

Sharing work

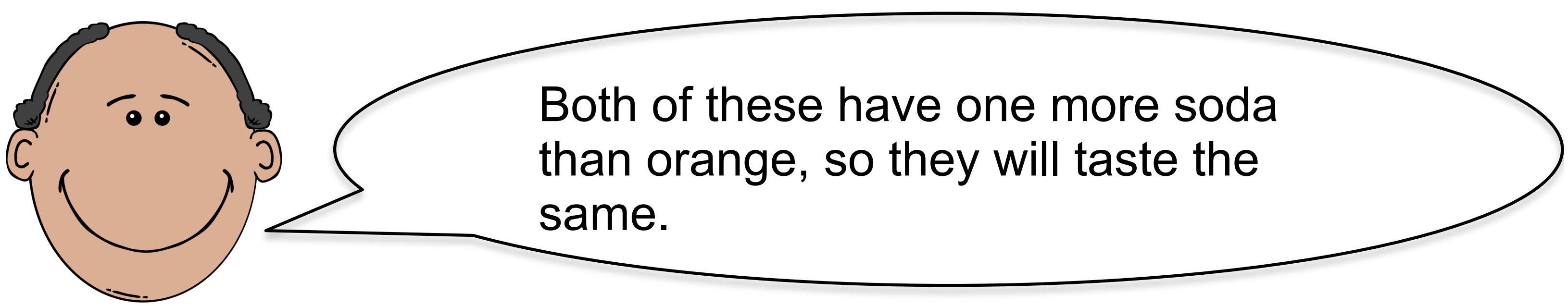
1. One person from each group get up and visit a different group.

2. If you are staying with your poster, explain your card order to the visitor, justifying the placement of each card.

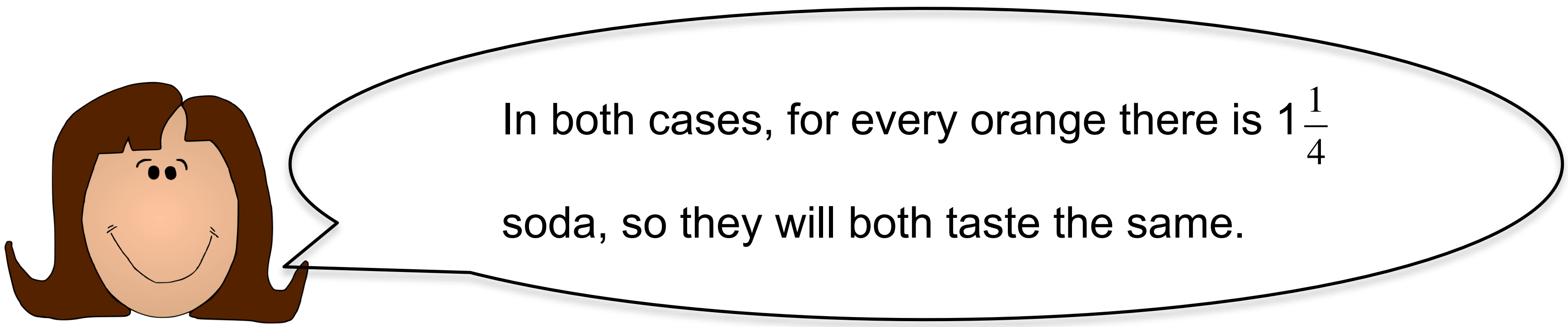
3. If you are the visitor, look carefully at the work and challenge any cards that you think are in the wrong place.

4. If you agree on the placement of the cards, compare your methods used when ordering.

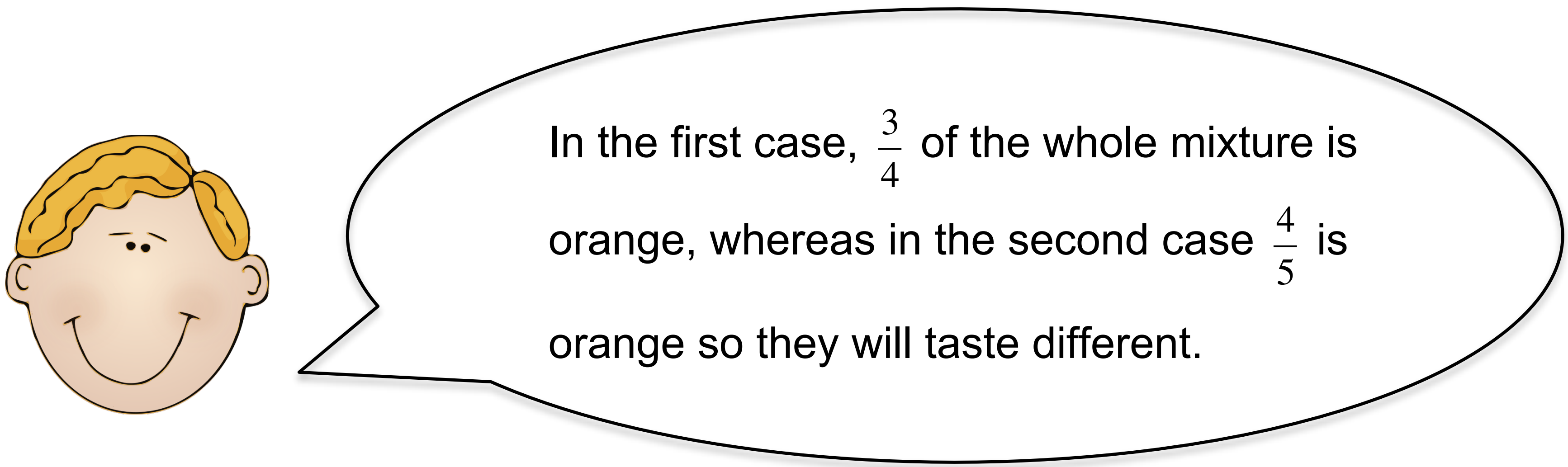
Emmanuel's Reasoning



Sifi's Reasoning



Alex's Reasoning



Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
Shell Center Team at the Center for Research in Mathematical Education
University of Nottingham, England:

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Hugh Burkhardt, Daniel Pead, and **Malcolm Swan** at the University of Nottingham

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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org