CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Evaluating Statements About Number Operations

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Evaluating Statements About Number Operations

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Understand the properties of number operations.
- Substitute integers, fractions, decimals, and negative numbers into inequality statements in order to test their validity.
- Represent inequalities algebraically and in words.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

6.NS: Apply and extend previous understandings of numbers to the system of rational numbers.

6.EE: Apply and extend previous understandings of arithmetic to algebraic expressions. Reason about and solve one-variable equations and inequalities.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 6 and 8:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

This lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task designed to reveal their current understanding and difficulties. You then review their responses and create questions for students to consider when improving their work.
- After a whole-class introduction, students work in small groups on a collaborative task, describing inequalities algebraically and in words.
- In the same small groups, students match descriptions of when the statements are true to their completed cards.
- To end the lesson there is a whole-class discussion.
- In a follow-up lesson, students work alone again on a similar task to the introductory task.

MATERIALS REQUIRED

- Each student will need a copy of the two assessment tasks: *Number Operations: Inequalities* and *Number Operations (revisited),* a mini-whiteboard, a pen, and an eraser.
- Each small group of students will need *Card Set A: Statements*, *Card Set B: When is the statement true?* (both sets cut-up), a large sheet of paper, felt-tipped pens, and a glue stick.
- There is a projector resource to support whole-class discussions.

TIME NEEDED

15 minutes before the lesson, a 100-minute lesson (or two 60-minute lessons), and 15 minutes in a follow-up lesson. Timings are approximate. Exact timings will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *Number Operations: Inequalities* (15 minutes)

Have students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.

Give each student a copy of the task *Number Operations: Inequalities*.

Read through the questions and try to answer them as carefully as you can.

It is important that, as far as possible, students are allowed to answer the questions without your assistance. Students should not worry too much if they cannot understand or do everything because in the next lesson they will work on a similar task, which should help them. Explain to students that by the end of the next lesson, they should be able to answer questions such as these confidently. This is their goal.

Number Operations: Inequalities

1. Abigail is thinking of a number:

If I subtract three from my number then my answer is less than four.

a. Could Abigail be thinking of 8? Explain your answer.

.....

.....

b. What numbers could she be thinking of? Describe them all.

.....

.....

c. Which of the following means the same as Abigail's statement? Check all that apply.

$x - 3 < 4$ ---------- $3 - x < 4$ ---------- $x - 3 > 4$ ---------- $3 - x > 4$ ----------

2. Benjamin is also thinking of a number:

If I multiply my number by four, then my answer is greater than ten.

a. Write down Benjamin's statement using algebra. .....

b. What numbers could Benjamin be thinking of? Describe them all.

3. Here are some more statements expressed in words and in algebra. The right hand column explains when the statements are true. Fill in the gaps. The first row has been done for you.

	Statement in words	Statement in algebra	What numbers would make the statement true?
Example	*When I add three to my number the answer is equal to seven.*	$x+3=7$	*Only the number 4 will make this true.*
(a)	*When I add four to my number the answer is greater than six.*	.....	.....
(b)	*When I multiply my number by six the answer is less than nine.*	.....	.....
(c)	Make up a statement to match the algebra:	$\frac{x}{3}<4$	.....
(d)	Make up a statement to match the algebra:	$7-x>6$	.....

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem solving approaches.

We suggest that you do not score students' work. The research shows that this will be counter-productive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or

- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students in the follow-up lesson.

Common issues	Suggested questions and prompts
Cannot translate between algebraic and written statements For example: The student re-writes Abigail's statement in words (Q1b) or is confused by 'less than'.	• What could you use instead of 'my number'? • Could you write the statement "I subtract three from my number and my answer is **equal to** four."
Treats subtraction as commutative For example: The student marks all alternatives given in Q1c or chooses $3 - x$ rather than $x - 3$.	• Is 'subtract three from' the same as 'subtract from three'?
Is unclear about the difference between equality and inequality For example: The student selects 7 as a value of x that satisfies Abigail's statement (Q1b).	• Is your answer less than 4? • What does the 'in' in the word 'inequality' stand for?
Does not fully understand the meaning of the signs <, > For example: The student writes $4x < 10$ and/or $4x = 10$ when the response should be $4x > 10$ (Q2a).	• What does Benjamin's statement tell us about the size of the answer? • Explain in your own words the meaning of the symbols: $<$, $>$, $=$.
Finds it difficult to substitute For example: The student chooses 8 as a value of x that satisfies Abigail's inequality (Q1a).	• Try replacing 'my number' with 'eight' in Abigail's statement. What comparison must you do then?
Uses only a restricted set of examples For example: The student selects the values of x that satisfy Abigail's inequality from the natural numbers only: 1, 2, 3, 4, 5, 6.	• Can you think of any other numbers that satisfy the inequality? • Are there any numbers between 6 and 7 that satisfy the inequality? • Are there any numbers less than 1 that satisfy the inequality? • Can you find the greatest or least number that satisfies the inequality?
Makes incorrect generalizations For example: The student states that numbers less than 3 satisfy Benjamin's inequality (Q2a).	• Can you try some numbers to show this? • What about fractions/decimals/negative numbers? What happens when you multiply these numbers by 4?
Completes the task correctly	• Can you show the range of possible answers using number lines?

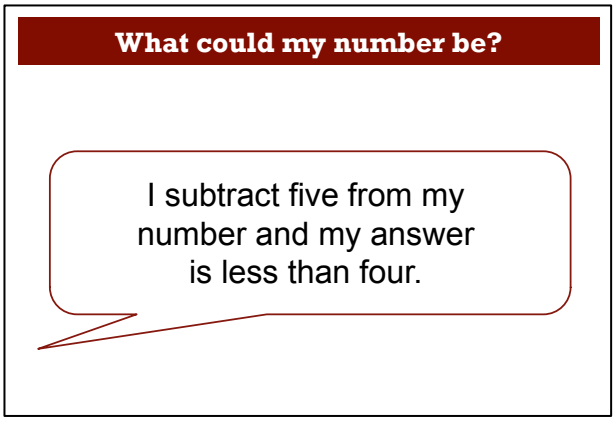
SUGGESTED LESSON OUTLINE

The work you have assessed will be returned to students after the lesson described below. Throughout the lesson encourage students to show their answers on a number line.

Whole-class introduction (20 minutes)

Give each student a mini-whiteboard, a pen, and an eraser.

Display Slide P-1 of the projector resource:



Spend some time discussing the numbers that satisfy the inequality.

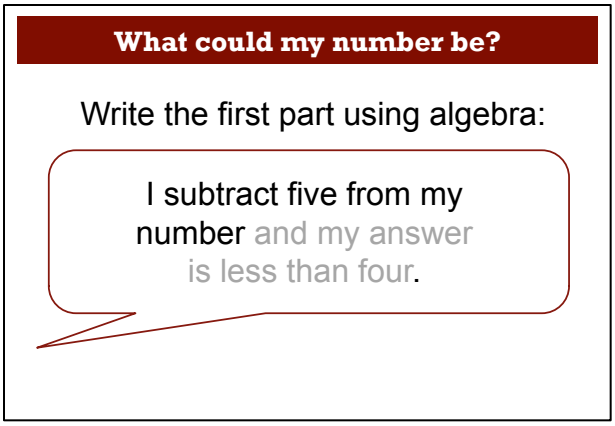
Try 10. What is 10 – 5? Is that less than 4?

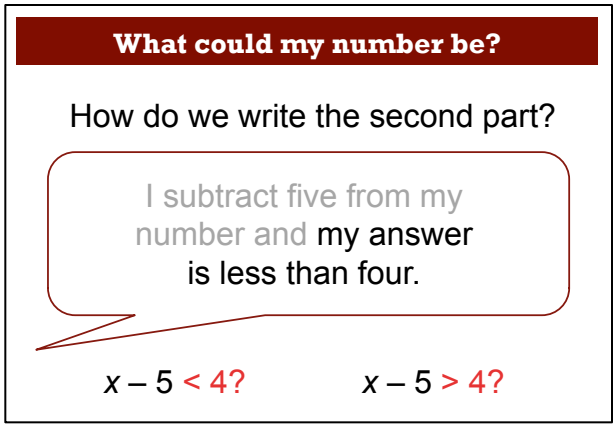
What number gives an answer less than 4?
*What **other** numbers give an answer less than 4?*

Can you use a number line to show all numbers that give an answer less than 4?

It is important that students recognize the meaning of 'less than' (or 'smaller than') and the range of values for x that will result. Note that both 'less than' and 'smaller than' are used commonly and so are 'more than', 'greater than' and 'bigger than'. We suggest you use these terms interchangeably to get the students used to the slightly different terminologies, as we have in these notes.

Display Slides P-2 and P-3 of the projector resource:





Use these slides as a basis for a discussion about

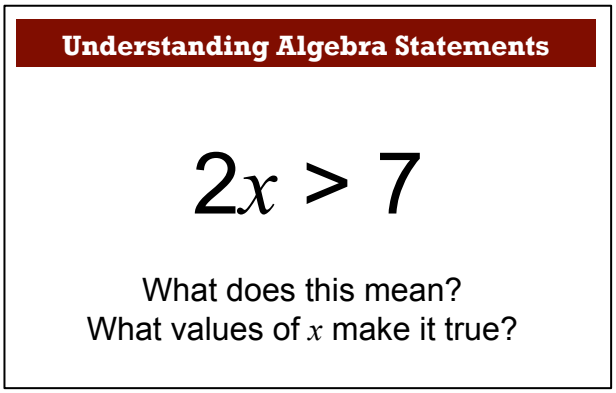
- representing 'my number' as x
- the fact that $x - 5$ is not the same as $5 - x$

- the direction of the inequality sign.

The purpose of this discussion is to further secure the students' understanding of translating between words and algebra, rather than attempting to solve the inequality.

Maximize participation by asking all students to show you their answers on their mini-whiteboards. If the students' understanding seems to be insecure, give additional examples.

Display Slide P- 4 of the projector resource:

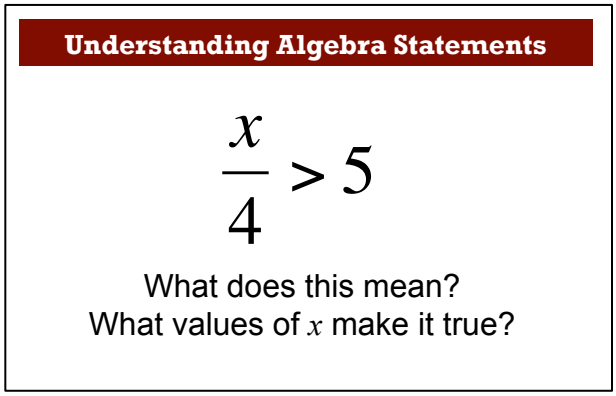


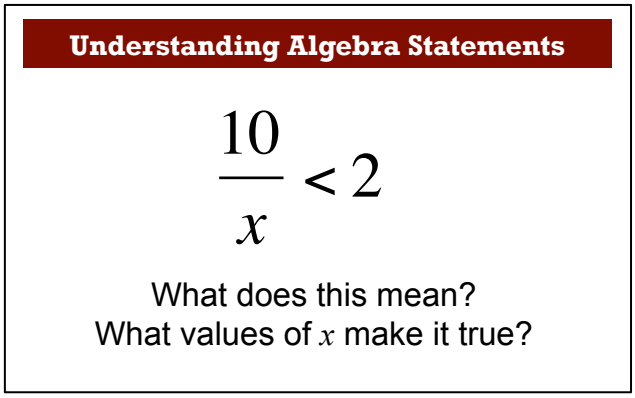
Here is an inequality. Can you say it in words?
Could x be 3? Could x be 4? What could x be?

Using your white boards, show me some values for x that make the statement true.
Now show me a smaller number that still makes it true….
What is the smallest number that makes it true? [There isn't one!]

Students should recognize that $x = 3.6$ makes the statement true but $x = 3.5$ does not. Discuss the answers the students offer and establish that x could take many (an infinite number!) of values.

Now show students Slides P-5 and P-6:





Repeat the above questioning to ensure that students are clear about both the division and the direction of the inequality. (They may not be sure whether the division is 'x divided by four' or 'four divided by x' for example.) Discuss the answers the students provide.

For $\frac{x}{4} > 5$, some students may suggest 20 and you might like to ask students:

*What is 20 divided by 4? Is the answer **greater** than 5?*
*What number will give an answer **greater** than 5?*
Can you show me the smallest number that will give an answer greater than 5? [Not possible!]

For $\frac{10}{x} < 2$, students may not spot that x can take values in two distinct ranges.

Is this true when x = 20? 10? 5? 1? -1?

Using your white boards, show me some more values for x that make the statement true. [x > 5 or x < 0.]

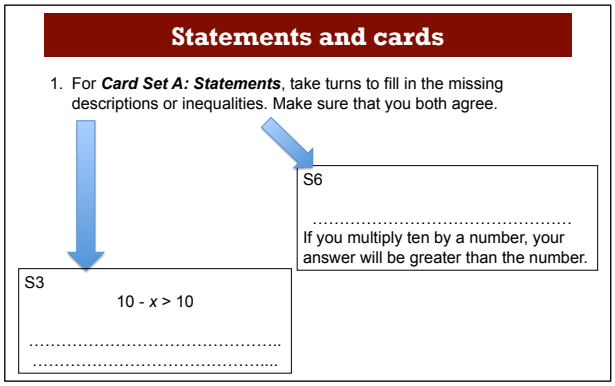
Discuss the answers the students provide. Some students may suggest five and you may point out that ten divided by five is equal to two and not smaller than two. Your students may also realize that negative numbers satisfy the inequality.

Can x be zero? Why not?
Can x be any negative number? What about negative one half? What about negative one quarter?

Collaborative small-group work: *Card Set A: Statements* (40 minutes)

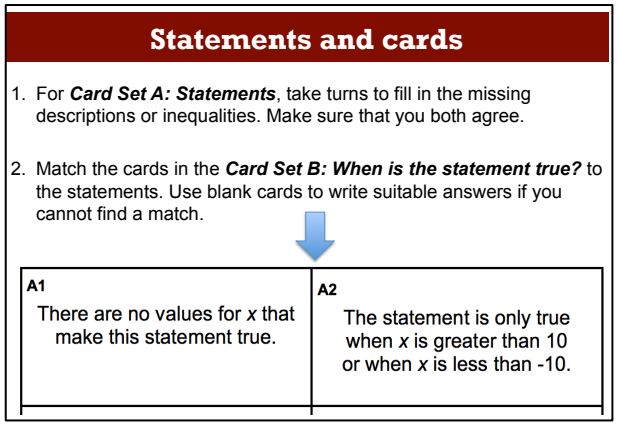
Organize the class into pairs and give each group the cut-up *Card Set A: Statements* and ask students to take turns to fill in the blanks.

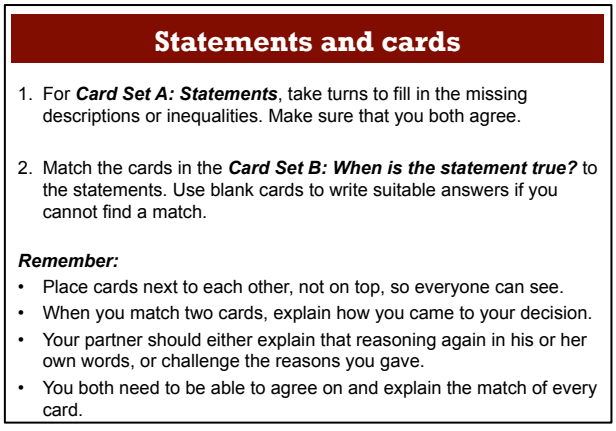
Slides P-7 of the projector resource summarizes what students should do:



Encourage students to justify their contributions and to make sure that both partners agree to an answer before it is written down. The main purpose of this activity is to consolidate their understanding of the relationship between the inequality expressed in words and in an algebraic form and the use of the inequality sign.

When students have completed the task give each pair the cut-up *Card Set B: When is the statement true?*, some poster paper, felt-tipped pens and a glue stick. Ask them to match the cards in this set to the *Statements* cards, following the instructions on Slide P-8 and P-9 of the projector resource:





The purpose of this structured work is to encourage each student to engage with their partner's explanations and to take responsibility for their partner's understanding.

In particular, point out that some of the cards are blank and students need to write suitable sentences on them when they find they have a statement that does not have a matching card. Once students are happy with a match they should stick their matched cards onto the poster paper, writing an explanation of their reasoning next to the cards.

While students are working in small groups you have two tasks: to note different student approaches to the task and to support student reasoning.

Note different student approaches

Listen and watch students carefully. Notice how students make a start on the task, where they get stuck, and how they overcome any difficulties. If students in the group take different approaches to completing the statement cards, encourage them to clearly explain the basis for their decision. Try to avoid giving students the information they need to fill in the cards. Instead, focus on helping them develop strategies for working, such as leaving statement cards they are struggling to complete until last and looking for differences and similarities amongst the statements.

When matching the cards with descriptions of when the statements are true, notice how students start. Do they begin with the statement cards or the descriptions of when they are true? Do they substitute fractions/decimals/negative numbers to check their assumptions? Do they provide examples and counter-examples?

Support student reasoning

Notice the quality and depth of students' explanations. Are students satisfied just to complete the statements and match the cards or do they give reasons for their choices? Do they challenge each other if they disagree on a card match? Encourage students to explain their reasoning carefully.

Jenny says that this statement is true for negative numbers. Ben, can you explain why?

How do you know that this statement is never true?
Can you convince me that it is true for NO value of x?

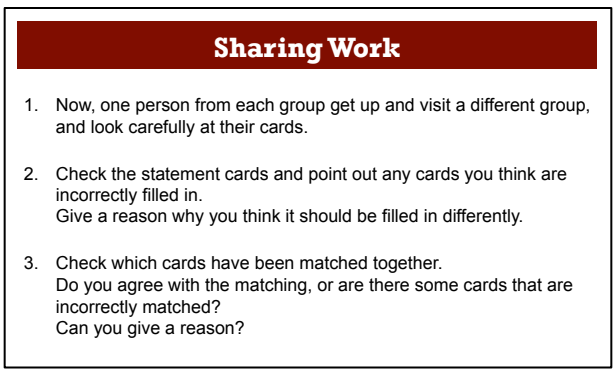
Can you show the range of values for x on a number line?

Extending the lesson over two days

If you are taking two days to complete the unit then you may want to end the first lesson here, ensuring that students have glued their matched cards onto their poster. Then, at the start of the second day, give students time to familiarize themselves with their own work before comparing posters with another group.

Sharing work (20 minutes)

As students finish matching the cards, get them to share their work. Use Slide P-10 to explain how they are to do this:



After a few minutes, ask students to reverse roles: the visitors should return to their own work and their partner should visit another group.

Once students have checked another group's cards, they may need to review both what they have filled in on their own cards and the card matching.

Whole-class discussion (20 minutes)

Once students have finished, organize a whole-class discussion about the cards. Begin by asking three or four pairs of students to present one pair of cards, explaining their completed *Statements* card and how they matched it with a *When is the statement true?* card. Focus the discussion on strategies:

How did you know what to put here?
How else can you explain that decision?

Which cards were difficult to fill in or match? Why was this?

Did any group make changes to their original matching after visiting another group?
Why was this?

When matching the cards, did you always start with a statement? Why was this?
Did anyone use a different strategy?

The focus of this discussion is to explore the processes involved in a range of different approaches.

Follow-up lesson: reviewing the assessment task (15 minutes)

Give each student a copy of the assessment task: *Number Operations (revisited)*, and their original scripts from the assessment task: *Number Operations: Inequalities*. If you have not added questions to individual pieces of work, then write your list of questions on the board. Students then select from this list only the questions they think are appropriate to their own work.

Read through your scripts from Number Operations: Inequalities and the questions [on the board/written on your script.] Answer these questions and revise your response.

Now look at the new task sheet, Number Operations (revisited). Can you use what you have learned to answer these questions?

Some teachers give this for homework.

SOLUTIONS

Assessment task: ***Number Operations: Inequalities***

1a. Abigail is not thinking of 8 because $8 - 3 = 5$ and 5 is not less than 4.

1b. Abigail could be thinking of all numbers smaller than 7.

1c. $x - 3 < 4$.

2a. $4x > 10$.

2b. Benjamin's statement is true when x is greater than 2.5.

3.

	Statement in words	**Statement in algebra**	**What numbers would make the statement true?**
(a)	*When I add four to my number the answer is greater than six.*	$x + 4 > 6$	$x > 2$ *All numbers greater than 2.*
(b)	*When I multiply my number by six the answer is smaller than nine.*	$6x < 9$	$x < 1.5$ *All numbers smaller than 1.5*
(c)	*When I divide my number by 3, the answer is less than 4.*	$\frac{x}{3} < 4$	$x < 12$ *All numbers less than 12.*
(d)	*When I subtract my number from 7, the answer is greater than 6.*	$7 - x > 6$	$x < 1$ *All numbers less than 1.*

Collaborative Activity: Completing Card Statements and Matching

S1 $10 + x > 10$ If you add a number to ten, your answer will be more than ten. A6 The statement is only true when x is greater than 0.	S2 $x + 10 < x$ If you add ten to a number, your answer will be less than the number. A1 There are no values for x that make this true.
S3 $10 - x > 10$ If you subtract a number from ten, your answer will be greater than ten. A3 The statement is only true when x is negative.	S4 $x - 10 > x$ If you subtract 10 from a number, your answer will be greater than the number. A1 There are no values for x that make this true.
S5 $3x > 10.5$ If you multiply three by a number, your answer will be greater than ten and one half. *The statement is only true when x is greater than 3.5*	S6 $10x > x$ If you multiply ten by a number, your answer will be greater than the number. A6 The statement is only true when x is greater than 0.
S7 $\frac{x}{10} < 2$ If you divide a number by ten, your answer will be less than two. *The statement is only true when x is less than 20.*	S8 $\frac{10}{x} < 10$ If you divide ten by a number, your answer will be less than ten. A5 The statement is only true when x is greater than one or when x is negative.
S9 $\sqrt{x} > 10$ The square root of a number is greater than 10. A4 The statement is only true when x is greater than 100.	S10 $x^2 > 100$ The square of a number is greater than one hundred. A2 The statement is only true when x is greater than 10 or when x is less than -10.

Assessment task: ***Number Operations (revisited)***

1a. Shamara couldn't be thinking of 8 because 10 – 8 = 2 is not greater than 2.

1b. Shamara's statement is written as $10 - x > 2$

1b. Shamara's statement is true when x is less than 8.

2.

	Statement in words	Statement in algebra	What numbers would make the statement true?
(a)	*When I add eight to my number the answer is greater than six.*	$x + 8 > 6$	$x > -2$ *All numbers greater than -2.*
(b)	*When I multiply my number by six the answer is less than six.*	$6x < 6$	$x < 1$ *All numbers less than 1.*
(c)	Make up a statement to match the algebra: *When I subtract 8 from my number the answer is greater than my number.*	$x - 8 > x$	*There are no values for x that make this true.*
(d)	Make up a statement to match the algebra: *When I divide 12 by my number the answer is less than 4.*	$\frac{12}{x} < 4$	$x > 3$ *All numbers greater than 3 or when x < 0.*

3. The two checked statements satisfy the condition: *"The statement is only true when x is greater than 1"*

$x + 1 < 2$		*True only when x < 1.*
$10x > 11$		*True only when x > 1.1*
$2 - x < 1$	√	*True only when x > 1.*
$\frac{8}{x} < 8$		*True only when x > 1 or when x < 0.*
$x + 1 > 2$	√	*True only when x > 1.*
$x^2 > x$		*True only when x > 1 or when x < 0.*

Number Operations: Inequalities

1. Abigail is thinking of a number:

If I subtract three from my number
then my answer is less than four.

a. Could Abigail be thinking of 8? Explain your answer.

---

b. What numbers could she be thinking of? Describe them all.

---

c. Which of the following means the same as Abigail's statement? Check all that apply.

$x - 3 < 4$	$3 - x < 4$	$x - 3 > 4$	$3 - x > 4$
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2. Benjamin is also thinking of a number:

If I multiply my number by four,
then my answer is greater than ten.

a. Write down Benjamin's statement using algebra. ……

b. What numbers could Benjamin be thinking of? Describe them all.

---

3. Here are some more statements expressed in words and in algebra.
The right hand column explains when the statements are true.
Fill in the gaps. The first row has been done for you.

	Statement in words	**Statement in algebra**	**What numbers would make the statement true?**
Example	*When I add three to my number the answer is equal to seven.*	$x+3=7$	*Only the number 4 will make this true.*
(a)	*When I add four to my number the answer is greater than six.*	……	…… …… …… ……
(b)	*When I multiply my number by six the answer is less than nine.*	……	…… …… …… ……
(c)	Make up a statement to match the algebra: …… …… …… ……	$\frac{x}{3}<4$	…… …… …… ……
(d)	Make up a statement to match the algebra: …… …… …… ……	$7-x>6$	…… …… …… ……

Card Set A: Statements

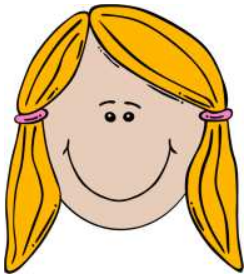
S1 $10 + x > 10$ If you add a number to ten, your answer will be more than ten.	S2 $x + 10 < x$ If you add ten to a number, your answer will be less than the number.
S3 $10 - x > 10$ …… ……....	S4 …… If you subtract ten from a number, your answer will be greater than the number.
S5 …… If you multiply three by a number, your answer will be greater than ten and one half.	S6 …… If you multiply ten by a number, your answer will be greater than the number.
S7 ……... If you divide a number by ten, your answer will be less than two.	S8 $\frac{10}{x} < 10$ …… ……...
S9 $\sqrt{x} > 10$ …… ……...	S10 $x^2 > 100$ …… ……...

Card Set B: When is the statement true?

A1 There are no values for x that make this statement true.	**A2** The statement is only true when x is greater than 10 or when x is less than -10.
A3 The statement is only true when x is negative.	**A4** The statement is only true when x is greater than 100.
A5 The statement is only true when x is greater than one or when x is negative.	**A6** The statement is only true when x is greater than 0.
A7	**A8**
A9	**A10**

Number Operations (revisited)

1. Shamara is thinking of a number:



If I subtract my number from 10
the answer is greater than 2.

a. Could Shamara be thinking of 8? Explain your answer.

b. How is Shamara's statement written using algebra?

c. What numbers could she have been thinking of? Describe all the possible numbers.

2. Here are some more statements expressed in words and in algebra.
   The right hand column explains when the statements are true.
   Fill in the gaps. The first row has been done for you.

	Statement in words	**Statement in algebra**	**What numbers would make the statement true?**
Example	*When I add three to my number the answer is equal to seven.*	$x+3=7$	*Only the number 4 will make this true.*
(a)	*When I add eight to my number the answer is greater than six.*	……	…… …… …… ……
(b)	*When I multiply my number by six the answer is less than six.*	……	…… …… …… ……
(c)	Make up a statement to match the algebra: …… …… …… ……	$x-8>x$	…… …… …… ……
(d)	Make up a statement to match the algebra: …… …… …… ……	$\frac{12}{x}<4$	…… …… …… ……

3. Fabio says:

The statement is only true when x is greater than 1.

a. Which of the following statements could Fabio be talking about? Check all that apply. Explain your reasoning next to each statement:

$x+1<2$	
$10x>11$	
$2-x<1$	
$\frac{8}{x}<8$	
$x+1>2$	
$x^2>x$	

What could my number be?

I subtract five from my number and my answer is less than four.

What could my number be?

Write the first part using algebra:

I subtract five from my number and my answer is less than four.

What could my number be?

How do we write the second part?

I subtract five from my number and my answer is less than four.

$x - 5 < 4$?

$x - 5 > 4$?

Understanding Algebra Statements

$$2x > 7$$

What does this mean?
What values of x make it true?

Understanding Algebra Statements

$$\frac{x}{4} > 5$$

What does this mean?

What values of x make it true?

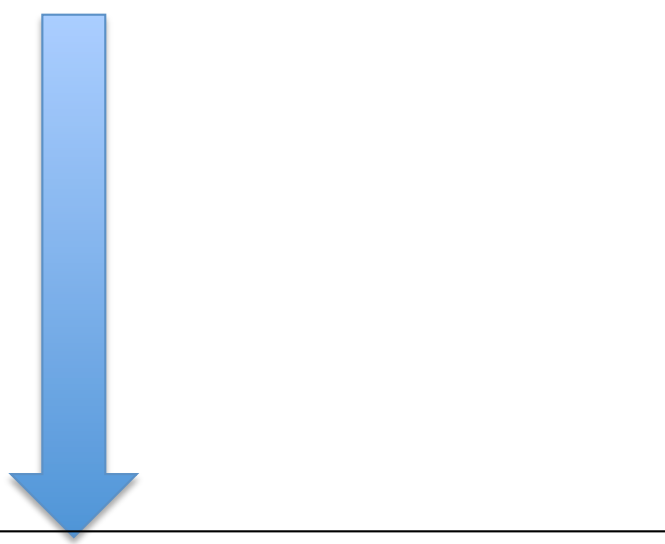
Understanding Algebra Statements

$$\frac{10}{x} < 2$$

What does this mean?
What values of x make it true?

Statements and cards

1. For ***Card Set A: Statements***, take turns to fill in the missing descriptions or inequalities. Make sure that you both agree.



S3

$10 - x > 10$

.....

.....

S6

.....

If you multiply ten by a number, your answer will be greater than the number.

Statements and cards

1. For ***Card Set A: Statements***, take turns to fill in the missing descriptions or inequalities. Make sure that you both agree.

2. Match the cards in the ***Card Set B: When is the statement true?*** to the statements. Use blank cards to write suitable answers if you cannot find a match.

A1 There are no values for x that make this statement true.	**A2** The statement is only true when x is greater than 10 or when x is less than -10.

Statements and cards

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2. Match the cards in the ***Card Set B: When is the statement true?*** to the statements. Use blank cards to write suitable answers if you cannot find a match.

Remember:

- Place cards next to each other, not on top, so everyone can see.
- When you match two cards, explain how you came to your decision.
- Your partner should either explain that reasoning again in his or her own words, or challenge the reasons you gave.
- You both need to be able to agree on and explain the match of every card.

Sharing Work

1. Now, one person from each group get up and visit a different group, and look carefully at their cards.

2. Check the statement cards and point out any cards you think are incorrectly filled in.
   Give a reason why you think it should be filled in differently.

3. Check which cards have been matched together.
   Do you agree with the matching, or are there some cards that are incorrectly matched?
   Can you give a reason?

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org