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COMMENTARY PAPER

Thoughts on scale

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Abstract This essay reflects on the challenges of thinking about scale—of making sense of phenomena such as continuous professional development (CPD) at the system level, while holding on to detail at the finer grain size(s) of implementation. The stimuli for my reflections are three diverse studies of attempts at scale—an attempt to use ideas related to professional development in two different countries, the story of how research did or did not frame a nationwide attempt at undergirding CPD, and a fine-grained study of the quality of a dozen mentors' implementation of CPD. The challenge is to "see the forest for the trees," to be able to situate such diverse studies within a larger framework. The bulk of this article is devoted to offering such a framework, the teaching for robust understanding (TRU) framework, which characterizes five fundamentally important dimensions of powerful learning environments. At the most fine-grained level, TRU applies to classrooms, establishing goals for instruction. But, more generally, it applies to all learning environments, and this characterizes important aspects of CPD. The TRU framework thus provides a unifying frame within which one can situate the studies in this volume.

Keywords Professional development · Scale · Powerful learning environments · Theories of proficiency

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1 The challenge

I begin with a metaphor to characterize the challenges that authors and readers face when confronting issues of scale. Consider the ways that map makers use insets—devices that allow for a focus on particular areas, while situating those areas within the larger whole. The use of insets allows for characterizations at two orders of grain size.

Now imagine a map of the world containing four insets. Two of those insets provide details about train service in a small number of towns in England and the United States. A third provides information about why bus lines are organized the way they are in Sweden, and the fourth describes the ways in which a dozen automobile drivers in Germany made use of new technology-enhanced automobiles. Your challenge, as a map-reader: use this information to think about the design of coherent transportation systems, worldwide.

That's quite a challenge. The point is that such insets, which serve to provide local descriptions of different aspects of a large and complex phenomenon, will only get one so far in understanding the issues at hand. The same is the case when one reads the chapters by Clark-Wilson, Hoyles, Noss, Vahey, and Roschelle (2015), Boesen, Helenius and Johansson (2015), and Kuzle and Biehler (2015), with an eye toward understanding the landscape of continuous professional development (CPD).

At the level of the individual papers, one can note interesting findings and point to aspects of the papers worth following up on, along various dimensions (e.g., with regard to theory, method, and practice). For example, Clark-Wilson, Hoyles, Noss, Vahey, and Roschelle's (2015) choice of methods—to use questionnaires, fleshing out the picture with two case studies—highlights a way to operate at two levels of scale. At the same time, there is much that the paper does

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not (and can not) do: there are serious issues relating to the meaningfulness and accuracy of questionnaire data (see, e.g., Schoenfeld, 2003), and the authors' choice of grain size means that it is difficult to learn much about the details and mechanisms of teachers' developmental trajectories.

The fact that Sweden embarked on a national program of professional development is in itself a marvel to someone from the United States, where teaching is not taken with the seriousness it deserves; Boesen, Helenius and Johansson's (2015) story of how research-based (or not!) Sweden's professional development programme turned out to be is interesting on its own. Here too, there are a thousand untold stories—e.g., the character of the research-to-policy linkages around the globe and, of course, the ways that such policies actually play out in practice.

Similarly, the paper by Kuzle and Biehler (2015) provides the details of a professional development course crafted by 12 mentors, with a focus on "deepen[ing] their professional knowledge of teaching statistics using digital tools, and to develop their competencies and knowledge for developing and implementing their own PD in statistics." Here we find, in an emergent way, some of the challenges of designing and implementing the PD. This too raises a host of questions. In what ways are the challenges similar to, or different from, the challenges of implementing non-technologically based PD? How much does knowledge matter, and in what ways does it matter? What else matters? And so on.

The landscape of professional development is huge and the three papers referenced above represent different aspects of three isolated parts of that landscape. To pursue the metaphor, the description of a vast landscape requires an atlas or its equivalent, with portrayals of that landscape at varying levels of grain size. An atlas allows one to take a distal view and to zoom in by degrees, with the understanding that the same degree of resolution (that is, the same grain size of detail) at different parts of the landscape produces information that can be seen as comparable, and linked to images of larger or smaller scale.

In terms of research, that means we need a set of frameworks (theories, where possible) for characterizing and situating the learning relevant to continuous professional development (CPD). With such a set of coherent and linked frameworks, we could begin to see how and where studies as disparate as the three studies in this section fit into the landscape; and we could begin to make coherent connections between the studies in this volume and the emerging literature on CPD.

2 A multi-level, recursive framework for examining learning, teaching, and PD

My purpose in what follows is to present and argue for a coherent way of viewing learning environments at various levels of scale, from the individual classroom to a national system. What I present is, of course, a mere sketch; a huge amount of detail remains to be filled in.

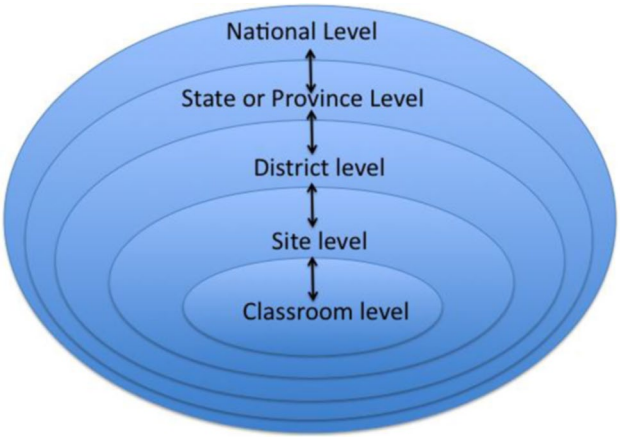


Fig. 1 Policy regarding professional development and its embodiment in practices and materials such as curricula, assessment, and professional development needs to be aligned, or the system works against itself

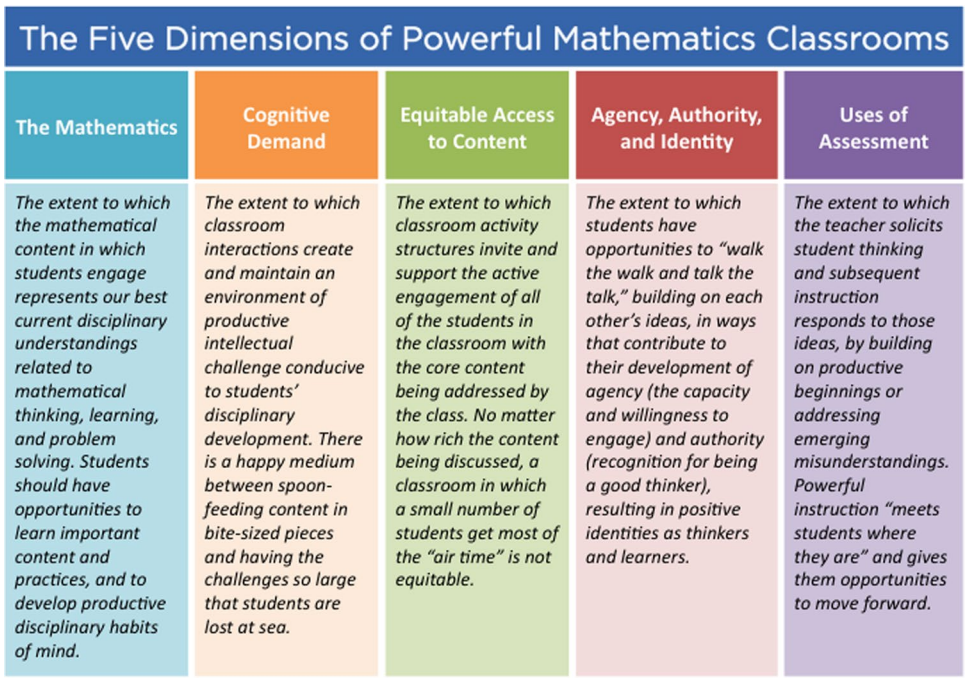
A first main point to understand is that for any system of professional development to function effectively, it must be *coherent*—the "messages" sent by various parts of the system must reinforce rather than contradict or undermine each other. There is clear evidence (see, e.g., Eisenhart, et al., 1993) that if teachers receive contradictory messages (e.g., the examinations they must give are focused on skills, while the rhetoric in professional development focuses on conceptual understanding) that the goals of professional development are undermined. Figure 1 describes the way things *should* be.

In what follows I begin at the classroom level, and then expand outward. The recursive nature of the framework I am about to discuss allows it to expand outward to fill Fig. 1.

2.1 What makes for powerful mathematics (and other) classrooms?

There is a substantial body of evidence (see, e.g., Schoenfeld, 2013, 2014; Schoenfeld, Floden, & the Algebra Teaching Study and Mathematics Assessment Project, 2014a, b) that a five-dimensional framework called Teaching for Robust Understanding of Mathematics, or TRU-math, serves to characterize the degree of richness of mathematics classrooms, and that a mathematics classroom that does well along these five dimensions will produce students who are powerful mathematical thinkers. See Fig. 2 for a top-level summary of the framework. There is reason to believe that this framing is complete (no further dimensions will be found to be essential), and that some level of

Fig. 2 The TRUmath framework: The five dimensions of powerful mathematics classrooms



quality along each of these dimensions is necessary to produce students who are effective mathematical thinkers and problem solvers.

Here is a brief overview of the framework. Extensive detail can be found at http://ats.berkeley.edu/tools.html and http://map.mathshell.org/materials/trumath.php. It goes without saying that the mathematics at the heart of classroom discussions must be rich; without that, there is no hope that students will emerge with a rich sense of the mathematics. But that is not enough. Most of us have been in classrooms where the exposition by the professor seemed crystal clear—but when students attempted the homework, it was clear that they had not understood very much at all. What matters is not only the content, but how students interact with it.

The remaining four dimensions of the TRU framework flesh out the interactions between the students and the mathematics. In fundamental terms, the second dimension (cognitive demand) is concerned with students' opportunity to engage productively with the mathematics—to do mathematical sense making, which comes about as the result of meaningful efforts, or what is also known as "productive struggle". The third dimension pertains to our definition of a powerful classroom, which is a classroom can only be considered powerful if it provides meaningful learning experiences for *all* students. This is the "access" or "equity" dimension. Fourth is the idea that powerful mathematics students are students who have productive mathematical dispositions. They see themselves as people who can engage productively with mathematics—that is, they have positive mathematical identities. Mathematics environments can either help students build such identities by providing them with opportunities to do and explain mathematics, or they can communicate the opposite by restricting their mathematical roles, minimizing sense-making opportunities, and either tacitly or explicitly demeaning anything save for the kinds of answers the teacher is looking for. (The statement "I'm just not a math person" from otherwise very intelligent people is the result of instruction that deprived students of agency). Finally, powerful instruction is flexible and makes effective use of formative assessment. It reveals what students know, and is adaptive, so that it can build on what the students understand and address problematic aspects of their knowledge.

It goes without saying that the summary description given in each column of Fig. 1 is merely a label; a substantial amount of work has to be done to unpack the contents of each dimension. Here, I will focus on dimension 1, the mathematics.

In effect dimension 1 asks the question "what does it mean to be a powerful or productive mathematical thinker?" The answer to that question shapes the goals of instruction. We have long known that mathematical proficiency involves far more than having mathematical knowledge: powerful mathematical thinkers work strategically, have good metacognitive skills, and have productive beliefs and habits of mind (Kilpatrick, Swafford, & Findell, 2001; Schoenfeld, 1985). Thus, the classroom must support the development of these attributes of mathematical proficiency.

To put this another way, what underlies dimension 1 is a *theory of mathematical proficiency.* Having a theory of

proficiency provides a set of goals for student learning, and a set of things to look for in the classroom treatment of mathematics. And, (cf. Fig. 1) actions at all levels of the system, from classroom activities through national policy, should be aligned with and supportive of this theory of mathematical proficiency.

As I begin the process of generalizing the teaching for robust understanding (TRU) framework, I make a few observations. First, the framework has been the object of a large amount of research (including research summaries and professional development tools) by the Algebra Teaching Study (see http://ats.berkeley.edu/, specifically the "tools" page) and the Mathematics Assessment Project (see http://map.mathshell.org/materials/index.php, specifically the "TRU Math Suite" page). The framework represents the distillation of the broad literature, and as such is firmly grounded in theoretical and empirical terms. There is, however, a substantial amount of empirical work to be done—for example to establish the relationship between measures of classroom performance using the TRU framework and student performance on a range of outcome measures, and to explore empirically the relative contributions of the five dimensions to students' mathematical performance and identities.

The first step in generalization is easy. It is only a small leap in generality to replace dimension 1 (understanding mathematics) with the content of another discipline (e.g., understanding physics, biology, history, literature, or computer science). Note that dimensions 2 through 4 are general and the research has shown that they apply across the boards. Thus, for example, if one wants to talk about powerful classrooms in computer science (TRU-CS), one can make the relevant changes to dimension 1 (guided by a theory of proficiency in computer science), along with appropriate computer science-specific modifications for dimensions 2 through 5. (Each dimension plays out in some different ways with regard to different disciplines. Positive domain identities in computer science are akin to, but different from, positive domain identities in mathematics or other fields). Thus, once can think of the "TRU-X" framework, where X represents any discipline. See Schoenfeld, 2014, for more detail.

2.2 What makes for powerful professional development? A framework for supporting teacher growth

Here, we make more of a leap in abstracting the framework. The research base on professional development is not as extensive or well fleshed out as the research base on classroom learning, so it cannot be distilled in exactly the same way as the research base on classroom learning was distilled to create for TRUmath and TRU-X. This section relies more on a plausibility case, which has very strong general backing. A primary warrant for what follows is the simple but essential observation that adult learners are still learners, and that powerful learning environments for adult learners should have the same properties as powerful learning environments for students (see, e.g., Bransford, Brown, and Cocking, 2000, pp. 26–27).

The key ideas behind the generalization are this:

(a) Being a proficient teacher, like being proficient at any profession, can be considered analogous to being proficient at any particular discipline. What matters is having a theory of proficiency to guide efforts at improvement.
(b) Learning environments can be distributed—they are a function of a community, which may convene in various ways (not simply inside a classroom).

That said, it is easy to abstract Fig. 2 to establish a framework within which to situate efforts at supporting professional development. Consider Fig. 3.

Before considering how this framework operates at the level of professional development, I note (cf. Fig. 1) that there must be strong alignment between the goals of Fig. 2 (for student learning) and Fig. 3 (for teacher learning). Happily, this is easy: a proficient teacher is one who (see dimension 1 of Fig. 3) creates powerful learning environments for students. Thus, the primary "content" goal of professional development in Fig. 3 is to support teachers in developing the skills, understandings, and habits of mind that allow them to create powerful learning environments for students, as reflected in the whole of Fig. 2.

When one considers professional development in general, the parallels to learning environments for students are clear. There is content, of course. That is dimension 1 (becoming an effective disciplinary thinker in Fig. 2, becoming an effective teacher of that discipline in Fig. 3). But, just as powerful classrooms are about students as learners and their interaction with content, powerful professional development is about teachers as learners and their developing capacity to teach in powerful ways.

We focus first on that first dimension—in effect, what it means to be an effective teacher. As above, what one needs to specify dimension 1 is a theory of proficiency. Theories of proficiency for teaching are not nearly as well fleshed out as theories of proficiency for mathematics students, but we do have the beginnings of such. Schoenfeld & Kilpatrick (2008) offer a provisional theory of proficiency in teaching mathematics, the main components of which are summarized in Table 1

The framework in Table 1 was developed some years before the TRU framework, but one can see the consistency between it and the aspects of student proficiency described in Fig. 2. A great deal needs to be done to flesh out what

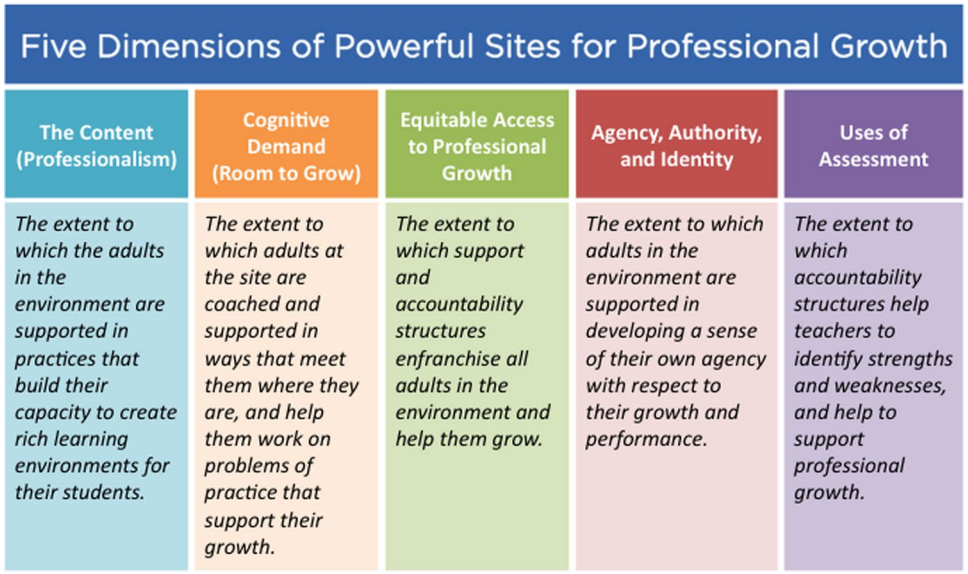


Fig. 3 The abstraction to professional growth: five dimensions of powerful adult learning environments

Table 1 A Provisional Framework for Proficiency in Teaching Mathematics Schoenfeld & Kilpatrick, 2008, p. 322

Knowing school mathematics in depth and breadth
Knowing students as thinkers
Knowing students as learners
Crafting and managing learning environments
Developing classroom norms and supporting classroom discourse as part of "teaching for understanding"
Building relationships that support learning
Reflecting on one's practice

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it takes to teach for proficiency, but this is a beginning, with respect to the "content" required for proficient teaching. The challenge, over time, is (a) to build a more refined theoretical base (including a sense of teachers' developmental trajectories) that characterizes teacher learning in the service of being able to create classrooms with the properties discussed in Fig. 2, and (b) to craft a set of tools that support teachers in that growth. Those tools might be considered the analogs of text and curricular materials for students.

Two examples of such tools are as follows. First, the Mathematics Assessment Project (MAP) offers 100 "Formative Assessment Lessons" or FALs designed to support the kinds of rich instruction that meets the aims of Fig. 2 (see http://map.mathshell.org/materials/index.php). The MAP project has produced 20 FALs for each of grades 6 through 10. These lessons are explicitly designed to support formative assessment regarding important content, and they are structured in ways that support all five of the dimensions in Fig. 2 (e.g., they provide opportunities for equitably supporting student discourse and class presentations, and for the diagnosis of student understandings in a way that allows for adjustment of cognitive demand). The explicit goal of the FALs' designers is that by having been scaffolded in the use of productive classroom techniques by these lessons, teachers will find it easier to use the same techniques in their "regular" lessons. Second, the MAP Project and the Algebra Teaching Study Project (see http://ats.berkeley.edu/) offer a set of tools including the TRU Math Conversation Guide (Baldinger & Louie, 2014) and the TRU Math Rubric (Schoenfeld, Floden, the Algebra Teaching Study and Mathematics Assessment Project (2014a, b), which are intended to support teachers, coaches, and learning communities in planning and reflecting on instruction, with a focus on enhancing instruction along the five dimensions in Fig. 2.

In a sense, these materials can be considered "texts and study guides" in the service of dimension 1 of Fig. 3—the "content" of being an effective teacher. In this way they can be seen as the analog of curricular materials for dimension 1 of Fig. 2—the "content" students should learn.

Dimensions 2 through 5 of the professional learning environment for teachers are every bit as important as dimensions 2 through 5 of the classroom learning environment are for students. Consider dimension 2. All learners, teachers included, need opportunities for sense making and for productive struggle—in the case of learning complex aspects of teaching, being able to take meaningful but

manageable steps on the way to crafting powerful learning environments for their students. The question is, where will teachers have the opportunities for being supported in sense-making and productive struggle? This is a challenge in the US, where teachers spend the vast majority of their time isolated in their own classrooms, with little opportunity for interaction with colleagues and even less opportunity for coherent, systematically planned development. (In that regard, things have changed very little in the almost 40 years since Dan Lortie (1975) wrote his classic study *Schoolteacher: A sociological study*). As other nations demonstrate, such isolation need not be the case. In Japanese lesson study (see, e.g., Fernandez & Yoshida, 2004; Lewis, Perry, Friedkin, Fisher, Disston & Foster, 2012; Lewis, Perry & Hurd, 2009) the goal is to provide ongoing opportunities for teachers' professional growth, under the assumption that it takes many years for even the most talented beginning teachers to develop into master teachers (and that learning is an ongoing, gradual process, no matter how experienced one may be). Interestingly, the lesson study environment supports teachers in many of the ways that powerful classroom environments support students. The environment is open—teacher enactment of lessons is visible and thus available for comment and improvement; but the commentary is supportive, in the service of improvement. That is part of a larger system of transparency, where teachers have a shared work environment, so that they can profit from each others' experience, and classrooms are open for inspection, so that teaching in the service of student learning is the basis of professional conversations. Such openness, along with opportunities for support and reflection, would be most welcome as mechanisms for teacher sense-making and productive struggle.

On dimension 3: I take it as axiomatic that a work environment should provide equitable opportunities for growth for all of the people who work in it. Inequitable access to opportunities for growth is as unacceptable in the workplace as it is in the classroom. Of course, providing equitable access to professional improvement is easier said than done. Just as there is "tracking" in US schools, a practice by which students are sorted into homogeneous classes according to their perceived "ability" or test scores (with the result, in general, that students in the low tracks are deprived of opportunities to engage in rich ways with the content), there is tracking in job placement in the US, where beginning or uncredentialed teachers are often placed in the most challenging school environments, with little opportunity for professional growth. The challenge for any district is to find ways to support *all* teachers with meaningful opportunities to develop as teachers, just as the challenge within the classroom is to provide *all* students with meaningful opportunities to learn.

On dimension 4: the people who do the best in any learning or work environment are those who have opportunities to develop an authentic sense of their emerging competence. Work environments that are structured to foster such engagement and growth are likely to support learners (in this case, teachers) in ways that contribute both to individual agency and to the overall enterprise. A teacher who is treated like a professional and given opportunities for growth, who is supported in developing deeper understandings of mathematics and student learning of mathematics, and who can see the results of this growth when his or her students become more powerful thinkers, is a teacher who will continue to grow. A teacher who is held accountable for student performance while provided little support for it may leave the profession in frustration.

Finally, on dimension 5: "meeting teachers where they are" is every bit as important as "meeting students where they are". The work environment that provides meaningful feedback about performance, and is geared toward improvement rather than critique, is the work environment most likely to support growth. Of course, that feedback should be grounded in observation tools that are consistent with the goals elaborated in dimension 1.

Note that we are dealing with systemic issues here. The relevant questions for an academic department within a school, for a school as a whole, for a school district, and beyond, are summarized in Table 2.

2.3 A prospective example

In the preceding discussion I have characterized some of the perspectives required for a coherent approach to CPD, and some tools that might help. Here I would like to suggest a way that a school or district could put them into play, in a coherent way. This is not purely hypothetical—we have plans to implement what I sketch below, if the political and funding climates permit.

Imagine that a school district has embedded a number of the Formative Assessment Lessons in the curriculum, or makes use of other rich curricular materials that provide affordances for student engagement along dimensions 1 through 5. Imagine as well that the administrative and coaching staff of the district have become fluent with the perspective, tools, and language of the TRU framework—so that they view instruction through that framework, and have access to tools (e.g., the TRU rubric to shape their classroom observations, and the TRU Conversation Guide to shape and support their conversations about the planning and implementation of lessons and professional development).

Imagine a mathematics department that has set aside time, on a regular and frequent basis, for conversations about teaching and learning. Early in the year, district coaches meet with the department to explain the framework, as a set of principles that support powerful classrooms. (This is easy to do. If a group watches a few sample videos and

Table 2 Issues for a department, school, or district to consider in shaping productive learning environments for teachers

1. What vision of teaching proficiency (dimension 1) guides policy regarding teacher performance and growth? How is that vision reflected in the structures that surround teaching, the support materials offered teachers (including curricula, tests, and PD materials) and the opportunities for professional growth? Are various structures and procedures (e.g., testing, presence or absence of professional development opportunities) aligned with that vision?
2. In what ways are teachers' environments structured so that teachers can engage in sensemaking and productive struggle (dimension 2) with regard to their teaching? Do they have opportunities to build a vision of what they want to achieve, and of what plausible next steps are? Do they have a framework for making progress, and support in doing so?
3. In what ways does the environment provide all teachers opportunities for growth, in a manner that can be characterized as equitable? (dimension 3)
4. In what ways does the environment provide opportunities for teachers to see themselves as professionals—to be focused on goals that matter, and to be recognized when they make steps toward achieving those goals? To what degree does the environment treat them as professionals? This is what fosters agency and productive identities (dimension 4)
5. In what ways does the environment provide meaningful feedback, in ways that teachers come to understand their strengths and areas for growth, and are provided support for progress? (dimension 5)

comments on what they notice, it is straightforward for an experienced coach to organize the group's comments by the five dimensions—showing that the framework is consistent with what the group itself has identified as important). The department head then volunteers to present a lesson plan at the next meeting, and have it vetted by the group, using the TRU Conversation Guide as a mechanism to inquire into the richness of the plan. He or she then teaches the lesson (open to all, and videotaped for those who have time conflicts) and, at a subsequent meeting, one of the coaches leads a debriefing session using the TRU framework as the organizing frame for the debriefing. The question is: how did things go, where can opportunities for students be made richer?

At that point, the department head asks for a volunteer—who's willing to be next for the same process? After a few rounds of such conversations, the department embarks on a form of lesson study, where the research questions for the lesson are derived from the TRU framework.

Our expectation is that after a few sessions, the language of the framework—rich mathematics, cognitive demand, sense making and productive struggle, equitable access, agency and identity, and formative assessment—will be familiar and easy to remember. The goal is then to have teachers both plan and reflect on lessons, using TRU as a frame. We hope that the communal support for such activities will create a productive environment focused on teacher and student growth, along the lines of Figs. 2 and 3. If these activities are iterated with enough frequency, we hope that the habits of mind they support will become internalized. Time will tell.

3 Discussion (coming full circle)

Let us continue the expansion implied by Fig. 1. My framing began at the classroom level (Fig. 2), and it has natural extensions at the department, school site, and district levels (all of which can be seen as represented in Fig. 3). But what takes place at those levels is shaped in fundamental ways by what takes place at the state and/or national levels. In the United States, for example, "high stakes testing" supported by federal funding plays a nationwide role in establishing and enforcing goals for both mathematics classroom teaching and for professional development; in Sweden, the federally supported *Boost for mathematics* professional development effort (Boesen, Helenius and Johansson, 2015) has shaped the evolution of teacher proficiency and presumably, teachers' classroom behavior. Thus, even if one is acting locally, one needs to think systemically.

The framework outlined in this article has both pragmatic and theoretical implications. On the pragmatic side, there is a clear argument that progress is best made in a system if goals and practices across all levels of the system are in synch. The implication of the research is that those goals and practices should be consistent with the frameworks outlined in Figs. 2 and 3. The challenge, then, is to support enough communication across levels to have a coherent system (with the right goals).

On the theoretical side, I return to the map metaphor that began this article. A major obstacle to both theoretical and empirical progress is that there has been little or no consistency across studies of different scale (and often across studies of the same scale)—different "insets" focusing on different things make it extremely difficult to put together the big picture. An atlas functions effectively because there is such consistency in its design: when one zooms in, the enlargements make sense vis-à-vis the larger and smaller levels of grain size. It seems to me that it would be possible build linked analytic frameworks such as those indicated in Figs. 2 and 3, at each of the levels from the classroom to the nation. Then, when studies of professional development (or anything else pertaining to the mathematics classroom) are conducted, they could be situated with regard to this interlocking framework. This would make it possible to conduct research that "adds up", where studies are framed in

Table 3 Issues for *any* learning environment

1. What vision of proficiency guides the activities in this environment? How is that vision reflected in what people do? Are structures and procedures within the environment aligned with that vision?
2. Do people in the environment have meaningful opportunities to make progress?
3. In what ways does the environment provide all participants opportunities for growth, in a manner that can be characterized as equitable?
4. In what ways does the environment support the development of agency and productive identities?
5. In what ways does the environment provide meaningful feedback to help people grow?

ways that can contrast with and/or build on others. In that way, we could begin to build a more coherent picture of the instructional and professional development landscapes.

Such coherence would be welcome. Indeed, one can see how such a framing would allow the field to situate studies in ways that they could be compared and built upon. Consider a theoretical "index" in which key words are linked to the framing given here, so that related studies could be identified and contrasted. Indeed, imagine a uniform framing for studies, in which authors are asked to identify which dimensions of the framework they address, at what levels of grain size. Clark-Wilson et al. (2015) could be more explicit about the developmental framework they employ, and point to the interaction between the constraints of the learning environment and the sustainability of desired practices; Boesen, Helenius and Johansson (2015) could situate themselves squarely as examining the contextual support for dimension 1, theories of CPD; and Kuzle and Biehler (2015) might frame their attempt at PD in ways that one could see more readily how their specific efforts were grounded in a theory of teaching proficiency and addressed issues of teachers' agency and identity, in ways that identified where they could be supported and that provided support in ways that engaged the teachers in sense making. By placing studies within such an overarching frame, it would be easier for the field to identify both contributions within and similarities/differences across research papers.

Finally, I do want to note that there is at least a plausibility case for the generality of the framework presented in Fig. 3. Consider any learning environment—work or family, for that matter. Every environment should be a learning environment. Table 3 provides a natural extension of the five questions given in Table 2.

I suspect these are useful questions for all learning environments, starting with classrooms and school sites, and expanding through all the levels indicated in Fig. 1.

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