PROBLEM SOLVING

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Deducing Relationships: *Floodlight Shadows*

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

For more details, visit: http://map.mathshell.org

Deducting Relationships: *Floodlight Shadows*

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to identify and use geometrical knowledge to solve a problem. In particular, it aims to identify and help students who have difficulty in:

- Making a mathematical model of a geometrical situation.
- Drawing diagrams to help with solving a problem.
- Identifying similar triangles and using their properties to solve problems.
- Tracking and reviewing strategic decisions when problem-solving.

COMMON CORE STATE STANDARDS

This lesson relates to **all** the *Standards for Mathematical Practices* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 1, 2, 3, 4, and 5:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

G-SRT: Prove theorems involving similarity.

INTRODUCTION

The lesson is structured in the following way:

- Before the lesson, students attempt an assessment task individually. You review their solutions and formulate questions to help them improve their work.
- At the start of the lesson students work individually answering your questions. They then work collaboratively in pairs or threes to produce a joint solution to the same task. They justify and explain their chosen method. Working in the same small groups, they critique examples of other students' work on the task.
- In a whole-class discussion, students explain and compare the alternative approaches they have seen and used.
- In a follow-up lesson, students work individually to reflect on what they have learned.

MATERIALS REQUIRED

- Each student will need a copy of the *Floodlights* task, the *How Did You Work?* questionnaire, and a sheet of squared paper.
- Each small group of students will need a large sheet of paper and copies of the *Sample Responses to Discuss*.
- Provide squared and plain paper, rules, pencils, protractors, and calculators for students to choose from. There is a projector resource provided to support whole-class discussions.

TIME NEEDED

30 minutes before the lesson, an 80-minute lesson (or two 45-minute lessons), and 15 minutes in a follow-up lesson. Timings are approximate. Exact timings will depend on the needs of the class.

BEFORE THE LESSON

Introduction: understanding shadows (10 minutes)

Begin by checking that students understand how shadows are formed. You could use a flashlight in a darkened room or display Slides P-1 and P-2 of the projector resource (*Shadows 1* and *Shadows 2*).

Suppose I turned on a flashlight in a dark room. What would happen? [A beam of light would shine from the flashlight.]

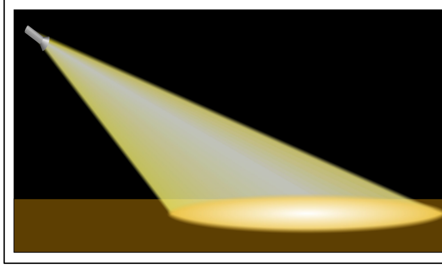
What would happen if someone stood in the beam of light? [The person would block the light, forming a shadow.]

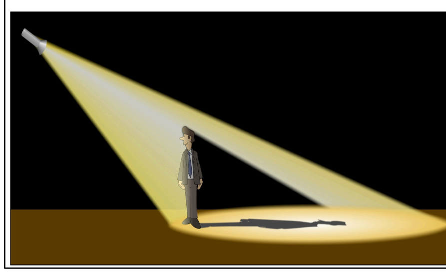
What is the shadow made of? [It is made of nothing! It's the place where light is not falling.]

Which part of the shadow is the greatest distance from the man's feet? [The shadow of his head.]

How would you measure the length of the shadow? [The distance from the man's feet to the furthest tip of darkness, rather than from the tip of the man's head to the furthest tip of the darkness.]

Today's task is to think about someone standing in a beam of light, casting a shadow. How does the length of his shadow change as he walks away from the light?





Assessment task: *Floodlights* (20 minutes)

Have the students do this task in class or for homework a day or more before the formative assessment lesson. This will give you an opportunity to assess the work and find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.

Give each student a copy of the task and some squared paper to work on. Briefly introduce the task and help the class to understand the problem and its context:

Floodlights

Eliot is playing football.

He is 6 feet tall.

He stands exactly half way between two floodlights.

The floodlights are 12 yards high and 50 yards apart.

The floodlights make two shadows of Eliot in opposite directions.

1. Draw a diagram to represent this situation.
   Label your diagram with the measurements.

2. Find the total length of Eliot's shadows.
   Explain your reasoning in detail.

3. Eliot walks in a straight line towards one of the floodlights.
   Figure out what happens to the total length of Eliot's shadows.
   Explain your reasoning in detail.

*In this problem, a football player stands half way between **two** floodlights.*

Each light throws a shadow.

Use the information to draw a diagram.

Think about what your diagram should show. Then read through the questions and answer them carefully.

Try to present your work in an organized and clear manner, so everyone can understand it.

It is important that, as far as possible, students are allowed to answer the questions without assistance.

Students who sit together often produce similar answers so that when they compare their work they have little to discuss. For this reason, we suggest that when students do the task individually, you ask them to move to different seats. At the beginning of the formative assessment lesson allow them to return to their usual seats. Experience has shown that this produces more profitable discussions.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem-solving approaches. The purpose of doing this is to forewarn you of issues that will arise during the lesson itself, so that you may prepare carefully.

We strongly suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this lesson unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students at the beginning of the lesson. You may also want to note students with a particular issue so that you can ask them about their difficulties in the formative lesson.

Common issues	Suggested questions and prompts
Does not understand how shadows are formed For example: The student fails to draw a line from top of floodlight to top of Eliot's head.	• Where are the shadows on your diagram? • How do floodlights create shadows? Can you show this on your drawing?
Draws a minimal diagram (Q1) For example: The student does not include all the information from the question. Or: The student labels the diagram incorrectly.	• How have you represented Eliot? • How have you represented the floodlights? • How have you represented … • Label your diagram clearly.
Chooses a scale drawing strategy but draws an inaccurate diagram (Q2)	• You are measuring to find the length of the shadows. What scale is your drawing? • How accurate does your diagram need to be?
Works unsystematically For example: The student draws two or three unconnected diagrams (Q3). Or: The student does not organize the information generated to show co-variation. Or: The student does not convert length measures to a common unit.	• Which are useful examples to draw? Why? • How can you organize your information so that you can make sense of the changes? • Which unit are you using for length?
Takes an unproductive approach (Q2, Q3) For example: The student unsuccessfully attempts to apply the Pythagorean Theorem.	• What are you trying to find out? Is your method helping you to get there? • How else could you approach the problem?
Unsuccessfully attempts to use similarity or trigonometry (Q2, Q3) For example: The student calculates incorrectly using ratios. Or: The student identifies missing lengths/angles but their relevance is not established.	• Which triangles are similar? How do you know? What else do you know about angles / triangles /…? • Which side of this triangle is a scaled version of side X? How do you know?
Uses an empirical method (Q3) For example: The student makes a scale drawing, or several scale drawings and measures them to find lengths.	• How will you extend your work to deal with all the different positions of the player? • What information do you have about angles / lengths / triangles in your diagram? What can you figure out?
Provides a poor explanation For example: The student explains calculations rather than giving mathematical reasons. Or: The student uses similar triangles without reference to similarity criteria.	• How can you convince a student in another class that your answer is correct? • You say these triangles are similar. How do you know?
Provides adequate solutions to all questions The student requires an extension task.	• Find a different way of tackling the problem to check your answer. • Figure out a way to solve the problem that will work whatever measures you are given. • What would happen if the player ran beyond one of the floodlights?

SUGGESTED LESSON OUTLINE

Individual work (10 minutes)

Return students' solutions to the *Floodlights* task and remind them of the problem.

Recall the work we did [last lesson] on shadows. Do you remember the task?

If you have not added questions to individual pieces of work, write your list of questions on the board now. Students can then select questions appropriate to their own work. Some teachers have found it helpful to provide students with a printed list of questions, highlighting those that apply to particular students.

I read your solutions and I have some questions about your work.

Spend 10 minutes, working on your own, answering my questions.

Collaborative work (20 minutes)

Organize the class into small groups of two or three students and give out a large sheet of paper to each group. Have a supply of equipment available for students who choose to use it (squared and plain paper, rules, pencils, protractors, and calculators).

Ask students to try the task again, this time combining their ideas.

Leave your individual work for now. I want you to work in groups.

Your task is to work together to produce a solution that is better than your individual solutions.

While students work in small groups you have two tasks: to note different student approaches to the task and to support students' problem solving.

Note different student approaches to the task

Observe students working with their chosen problem solving approaches. Note their mathematical decisions. Do they choose scale drawing, similar triangles, or try to use the Pythagorean theorem? Which resources do they ask for? Do they notice if they have chosen a strategy that does not seem to be working? If so, what do they do?

Do they try to use scale drawings? If so, are the drawings accurate? How do they try to adapt their method for Q3? Do they work systematically? Do they change approach?

Do students use similar triangles? Which triangles do they identify as similar from their diagrams? How do they justify their claims of similarity? Does their approach work for Q3?

Support student problem-solving

Try to avoid making suggestions that direct students towards a particular approach at this stage. Some students prefer to use scale drawings rather than take an analytic approach. Students can learn a great deal from trying out unfruitful methods (e.g. the Pythagorean Theorem) and discussing why these approaches do not work.

Ask questions that help students to clarify their thinking. You may find it helpful to use some of the questions in the *Common issues* table. If several students in the class are struggling with the same issue, you could write a relevant question on the board. You might also ask a student who has performed well on a particular part of the task to help a struggling student.

If students find it difficult to get started, these questions might be useful:

What do you already know? What do you need to know?

How can you show this information in a diagram?

What shapes do you see in your diagram?

Are there any construction lines you could add? How do they help?

What do you already know about triangles/angles? What else? Write it all down.

Now think about what you know about these triangles/angles.

Ask each group of students you visit to review their state of progress:

Review your work so far.

What was your strategy for solving this problem?

What work have you been doing?

What do you know now that you did not know before?

What have you learned so far that will help you solve the problem?

Are you going to continue with this strategy?

Are there any other approaches you could try?

In trials we found that many students located sides of triangles that they did not really need. Prompting students to monitor their work in this way will help them become more effective and independent problem solvers.

It is important that you ask the review questions of both students who are, and are not, following what you know to be productive approaches, whether or not they are stuck. Otherwise, students will learn that your questions are really a cue to switch strategy!

Prompt students to use clear and accurate language. It may help to prompt students to use labeling and notation, so they are able to refer to lengths and angles without saying 'this' and 'that'. In particular, students may make vague reference to 'similar sides' or 'proportion'. Clarifying the language can help students identify which particular proportional relationship they need to work with in calculations and help make their reasoning more rigorous and convincing.

Whole-class discussion: sharing methods (10 minutes)

Ask two or three groups to share their general ideas for approaching the task. Select groups that have different ideas and invite them to share these. It does not matter if students have not quite finished.

I would like a few of you to share your ideas for tackling the problem.

I don't want you to tell us the answers, but just give us some idea of the approach that you are finding most useful.

Extending the lesson over two days

If you are taking two days to complete the lesson unit then you may want to end the first lesson here. At the start of the second day, allow students time to familiarize themselves with their joint solution before moving on to the collaborative analysis of sample responses.

Collaborative analysis of *Sample Responses to Discuss* (20 minutes)

Give each group of students a copy of each of the three *Sample Responses to Discuss*. This task gives students the opportunity to evaluate possible approaches to the task without providing any complete solution strategy. Wendy's approach uses scale drawing, while both Tod and Uma use similar triangles, but in different ways.

Explain the task. Specific questions are given on the *Sample Responses to Discuss*.

I'm giving you some work on this problem written by students in another class.

None of the solutions are completely correct.

Work together on one student's solution at a time.

Answer the questions below each sample response, explaining your answers clearly.

Slide P-4 of the projector resource summarizes these instructions.

During the small-group work, support the students in their analysis. As before, try to help students develop their thinking, rather than resolve difficulties for them. Note similarities and differences between the sample approaches and those the students took in the small-group work.

Whole-class discussion: *Sample Responses to Discuss* (20 minutes)

Organize a whole-class discussion to consider issues arising from the analysis of *Sample Responses to Discuss*. You may not have time to address all these issues, so focus your class's discussion on the issues most important for your students, using what you noticed while observing students' work. Slides P-5, P-6, and P-7 can be used to support this discussion.

Focus the discussion on the strengths and weaknesses of the different solution methods:

Which approach did you like best? Why?

Which approach was most difficult to understand? What was difficult about it?

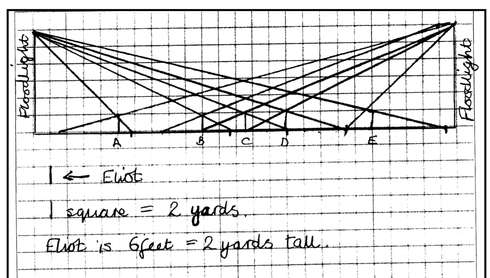
Which methods would be easiest to use if the measures changed? For example, if the person's height was changed, or the distance between the floodlights was changed?

Which method helps us understand why the total shadow length stays the same?

The following commentary on the three pieces of work may help you to prepare for the discussion:

Wendy has tried to solve the problem by scale drawing. The scale of the drawing is appropriate for the problem. However, her inaccurate lines, blunt pencil, and rough readings of measures introduce error into her data. Wendy organizes her data well, working systematically through a range of positions for Eliot and recording the data in order.

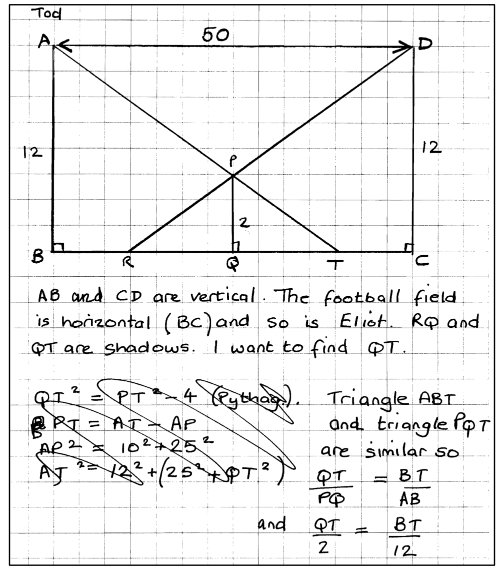
Wendy's model is simple and could be used to produce an accurate solution. However, it is only a descriptive model; used accurately, she could discover that the total length of the shadows is constant, but this would not give insight into **why** that is the case. Even if it were accurate, Wendy would only have produced an inductive solution to Q3, rather than a proof of the result. Furthermore, she would need to make a completely new drawing were she to try to generalize to different measures of player, heights of and distance between floodlights.



Place Eliot stands	Left shadow	Right shadow	Total Shadow
A	$3\frac{1}{2}$	1	$4\frac{1}{2}$
B	$2\frac{1}{2}$	$1\frac{1}{2}$	4
C	$2\frac{1}{2}$	$2\frac{1}{2}$	~~4~~ 5
D	2	3	5
E	$1\frac{1}{2}$	$4\frac{1}{2}$	6

As Eliot goes to the right, the left shadow gets shorter and the right shadow gets longer. Total shadow stays about the same.

Tod's method relates to Q2. He makes a minor error in his first statement: Eliot is not horizontal!



Tod is helpful in telling us what he is trying to find.

Tod begins by using the Pythagorean Theorem. After a few lines he realizes that this approach is going to get very messy (when he realizes that AT must be expressed in terms of QT). He therefore abandons it and uses similar triangles.

Tod does not explain how he knows triangles ABT and PQT are similar. You might ask students for a more rigorous argument.

This method may be continued to obtain:

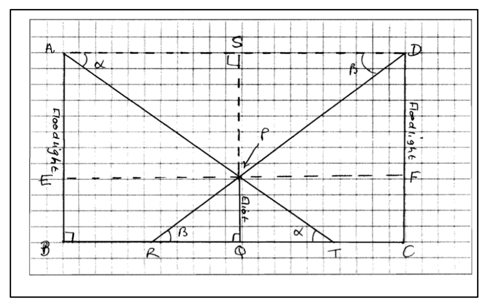
$$\frac{QT}{2} = \frac{QT + 25}{12}$$
$$\Rightarrow QT = 5$$

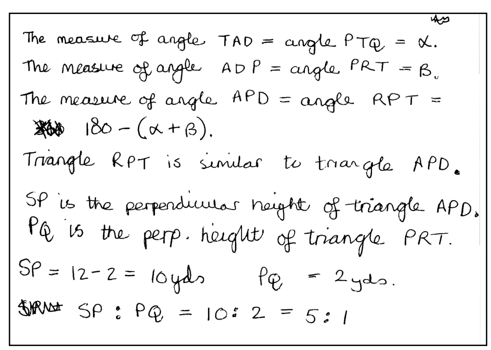
So one shadow is 5 yards.

By a similar argument the other is also 5 yards, so the total shadow length is 10 yards.

This method needs to be revised for Q3, with 25 replaced by a variable.

Uma provides a solution method for the more general Q3. Her diagram is not to scale, but does not need to be. She adds construction lines to her diagram and notes equal angles but does not justify those claims.




The measure of angle TAD = angle PTQ = α.
The measure of angle ADP = angle PRT = β.
The measure of angle APD = angle RPT =
180 − (α + β).
Triangle RPT is similar to triangle APD.
SP is the perpendicular height of triangle APD.
PQ is the perp. height of triangle PRT.
SP = 12 − 2 = 10yds PQ = 2yds.
SP : PQ = 10 : 2 = 5 : 1

She could have found angle APD = angle RPT more directly had she recognized that these are opposite angles.

Uma has chosen to place the line SQ so that it is almost central on her diagram. That the length BQ is arbitrary ensures a general solution; this would be clearer in Uma's diagram were SQ less centrally positioned.

Uma claims that triangle RPT is similar to triangle APD but does not justify this.

She notes that SP is the perpendicular height of triangle APD and that PQ is the perpendicular height of triangle PRT and finds the ratio of the lengths SP: PQ.

Uma's solution is incomplete. She needs to explain **why** the ratio of the sides in similar triangles is the same as the ratio of the altitudes of those triangles. Students may just quote this fact, or may show that SPD is a triangle similar to triangle RQP and triangle ASP is similar to triangle PTQ.

AD is fixed. Therefore, RT is fixed in length: the total length of the shadows does not vary with Eliot's position on the line between the two floodlights.

Uma's is the most elegant of the solution methods and the most general. It is a clear analytical and explanatory model: it shows why the total length of the shadows is constant.

Finally, bring the discussion back to the students' own problem-solving work.

Think back to last lesson and the beginning of this lesson. How did you decide which math to use?
Did you choose a helpful approach?

Did you, like Wendy, try specific cases until you got a good 'feel' for the problem?

Did you, like Tod, change your approach half way through because you didn't think it was working? (Using unproductive strategies is a natural part of problem solving! You need to learn to expect this.)

Did you, like Uma, try to solve the problem for the most general case?

Follow-up lesson: individual review (15 minutes)

Give each student a copy of the *How Did You Work?* questionnaire.

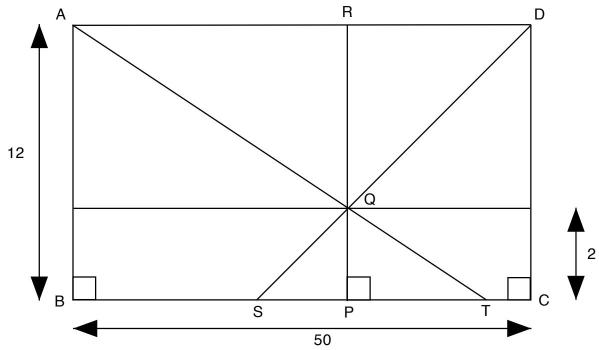
Think carefully about your work on this task. On your own, answer the review questions as carefully as you can.

You may also like to invite students to produce a fresh, complete and correct solution to the problem using one of the methods discussed in the lesson.

Some teachers give this as a homework task.

SOLUTIONS

There are many ways of answering the floodlights problem as the lesson notes show. Below is a method for the general solution:



The floodlights are positioned at A and D, 12 yards vertically above the ground BC and 50 yards apart.
The line PQ is a mathematical model for the player, 2 yards tall. Assume he is standing vertically. The shadows made by the floodlights are at PS and PT.

Since AB and CD are equal and vertical, then ABCD is a rectangle. AD is therefore parallel to BC.

Look at the right hand shadow PT.

First we show that triangle ARQ is similar to triangle TPQ.
AT is a transversal for the two parallel lines AD and BC, so <QAR = <QTP.
<TPQ = <ARQ = 90°
As two angles in APQ and TPQ are equal, then the triangles are similar.

Thus: $\frac{PT}{AR} = \frac{PQ}{QR} = \frac{1}{5}$, so the shadow length PT is one fifth of the length AR. (1)

Look at the left hand shadow PS

By an exactly analogous argument we know that triangle DRQ is similar to triangle SPQ. (Just repeat the above reasoning replacing A with D and T with S).

Thus $\frac{PS}{DR} = \frac{PQ}{QR} = \frac{1}{5}$, so the shadow length PS is one fifth of the length DR. (2)

Combining (1) and (2), we see that:

$$PT + PS = \frac{1}{5}AR + \frac{1}{5}DR = \frac{1}{5}(AR + DR) = \frac{1}{5}AD = 10$$

So this leads to the surprising result that the total shadow length is always 10 yards, wherever the player stands between the two floodlights.

Floodlights

Eliot is playing football.

He is 6 feet tall.

He stands exactly half way between two floodlights.

The floodlights are 12 yards high and 50 yards apart.

The floodlights give Eliot two shadows, falling in opposite directions.

1. Draw a diagram to represent this situation.

   Label your diagram with the measures.

2. Find the total length of Eliot's shadows.

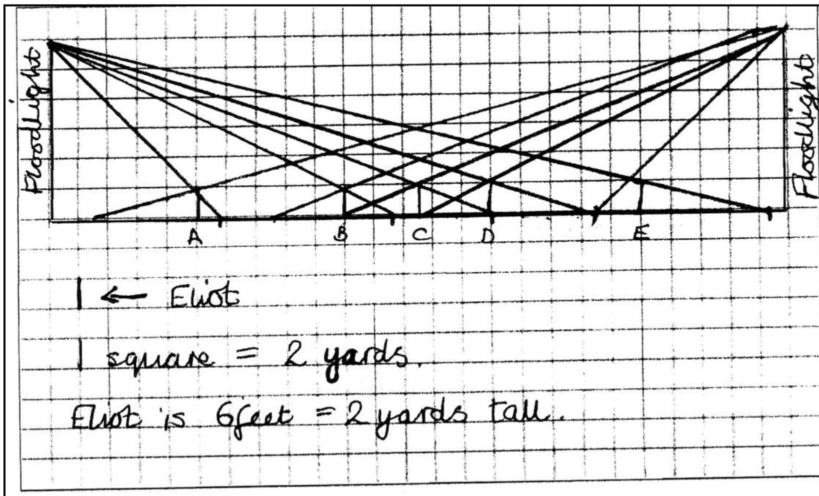
   Explain your reasoning in detail.

3. Suppose Eliot walks in a straight line towards one of the floodlights.

   Figure out what happens to the total length of Eliot's shadows.

   Explain your reasoning in detail.

Sample Responses to Discuss: Wendy



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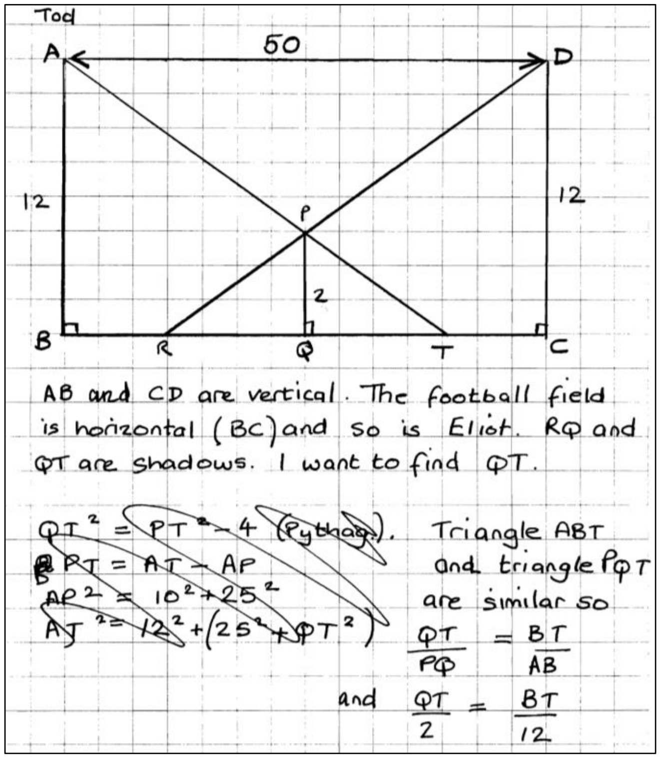
As Eliot goes to the right, the left shadow gets shorter and the right shadow gets longer. Total shadow stays about the same.

1. Explain what Wendy has done.

2. How could Wendy's work be improved?

Sample Responses to Discuss: Tod



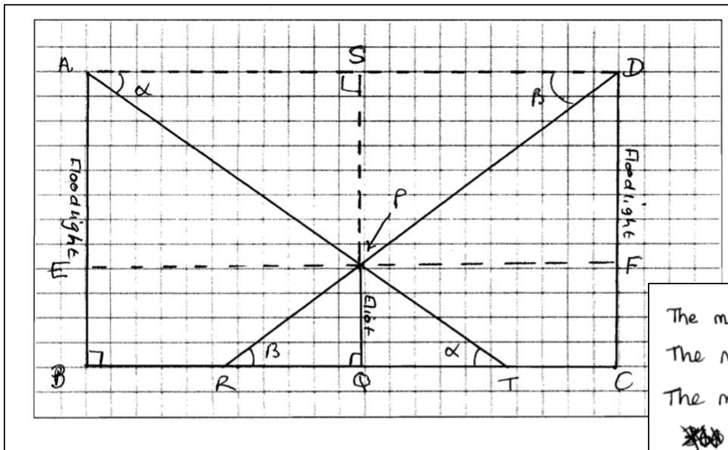
1. Explain what Tod has done.

2. How could Tod's work be improved?

3. Tod crossed out work using the Pythagorean Theorem. Why did he do this?

4. Try to complete Tod's work to get a solution.

Sample Responses to Discuss: Uma



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1. Explain what Uma has done.

2. How could Uma's work be improved?

3. Try to complete Uma's work to get a solution.

How Did You Work?

1. Check (✓) any math you used when completing the *Floodlights* task:

	Measuring lengths		Calculating the size of angles
	Drawing a diagram		Finding similar triangles
	Using the Pythagorean Theorem		Making a scale drawing
	Proving angles were congruent		Measuring angles
	Producing a results table		Showing triangles are congruent

Any other math you used:

2. Say which solution you prefer. (Check (✓) one box.)

	I like my own solution best.
	I like our group solution best.
	I like my own solution and our group solution the same.

Explain your answer:

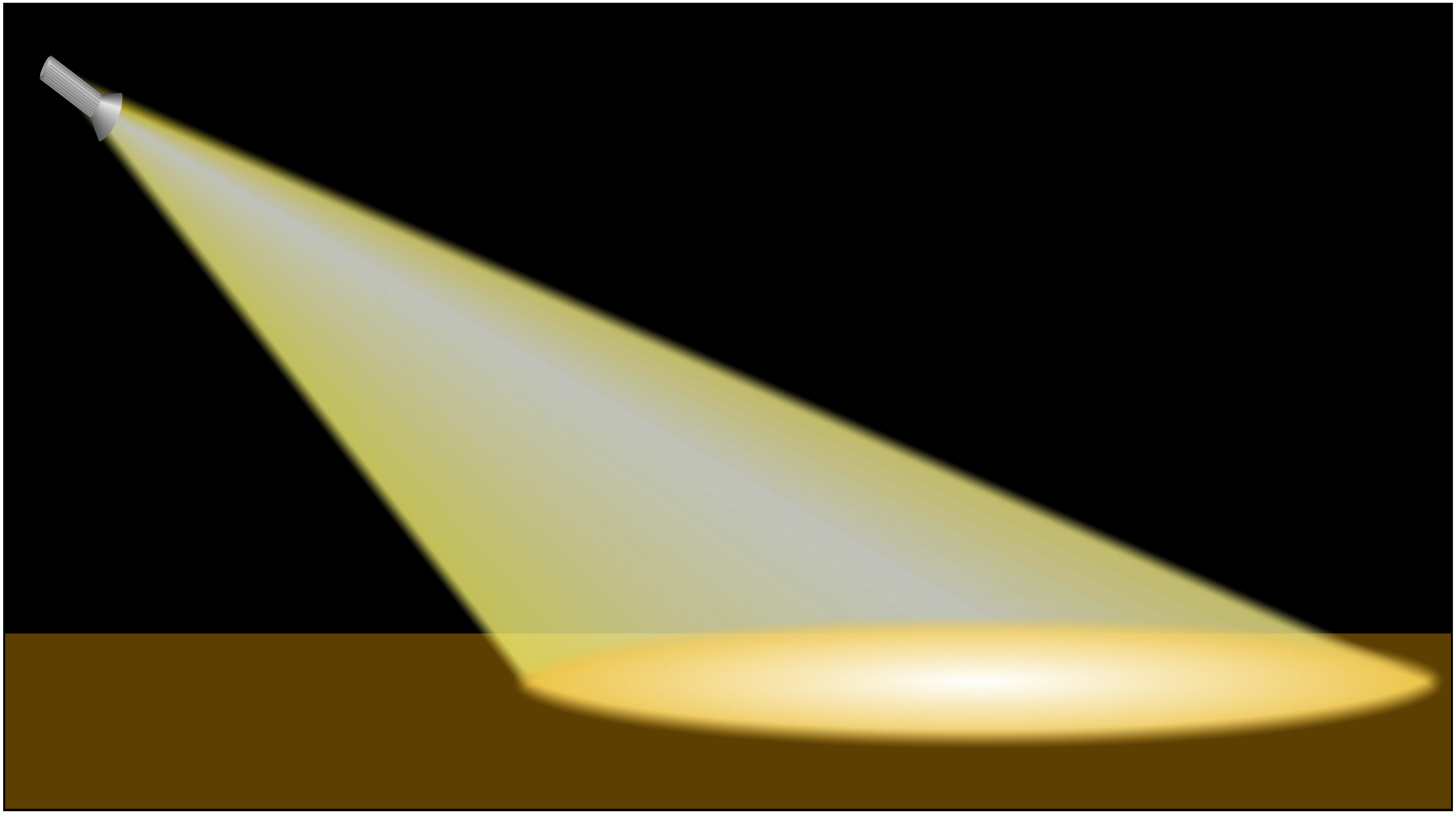
3. Check (✓) the *Sample Response* most like the method your group used.

	Our method was like Wendy's method		Our method was like Tod's method
	Our method was like Una's method		None of the methods were like our method

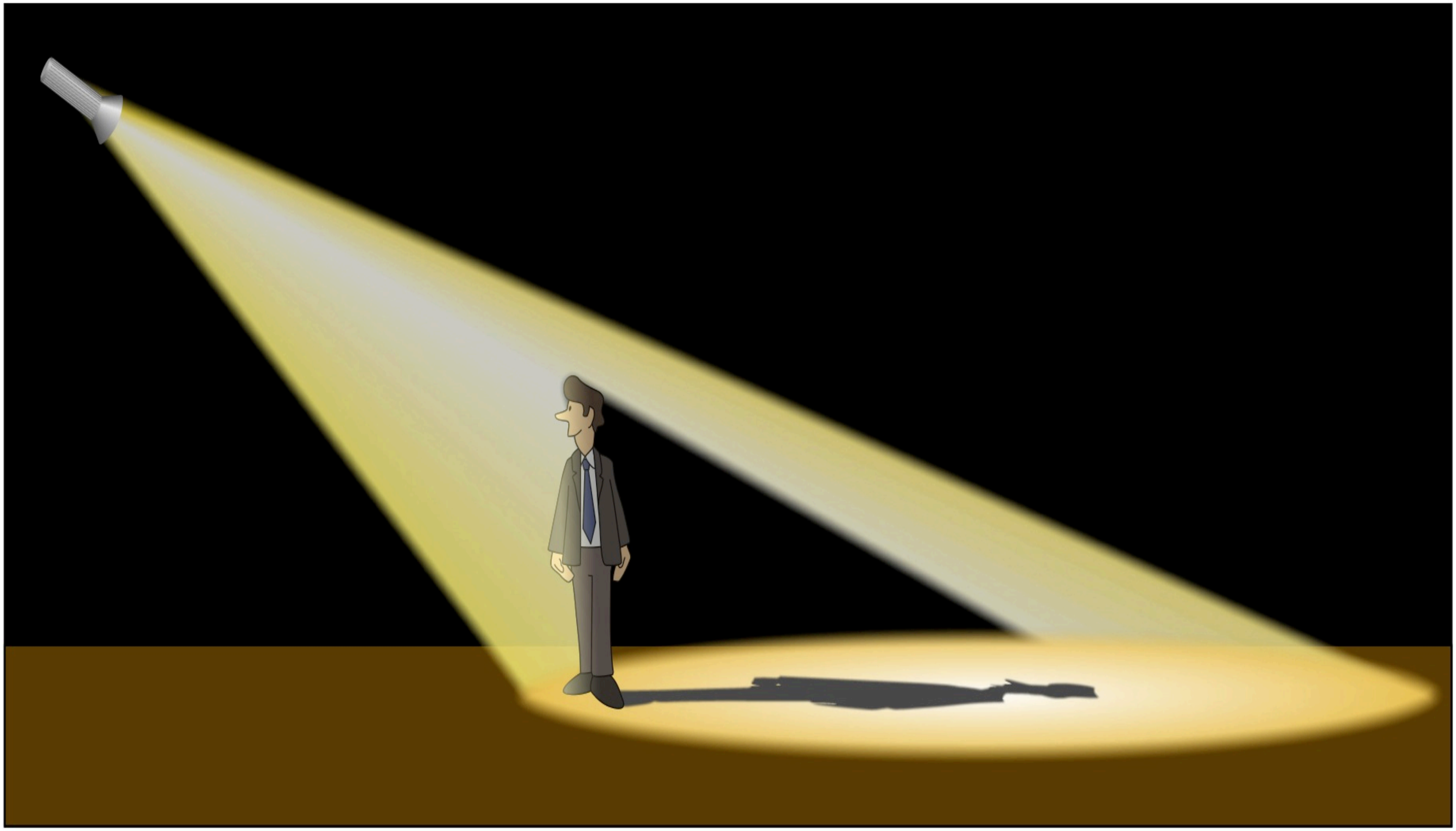
Which method do you think is best? Why?

4. What advice would you give to a student about to study the *Floodlights* problem?

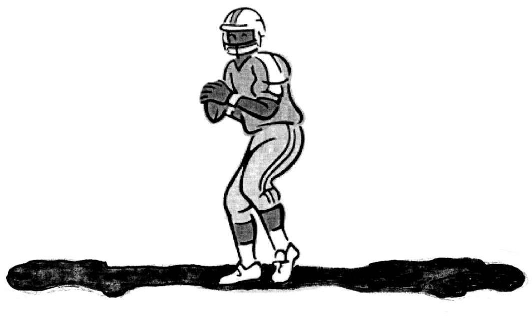
Shadows 1



Shadows 2



Floodlights



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Analyzing *Sample Responses to Discuss*

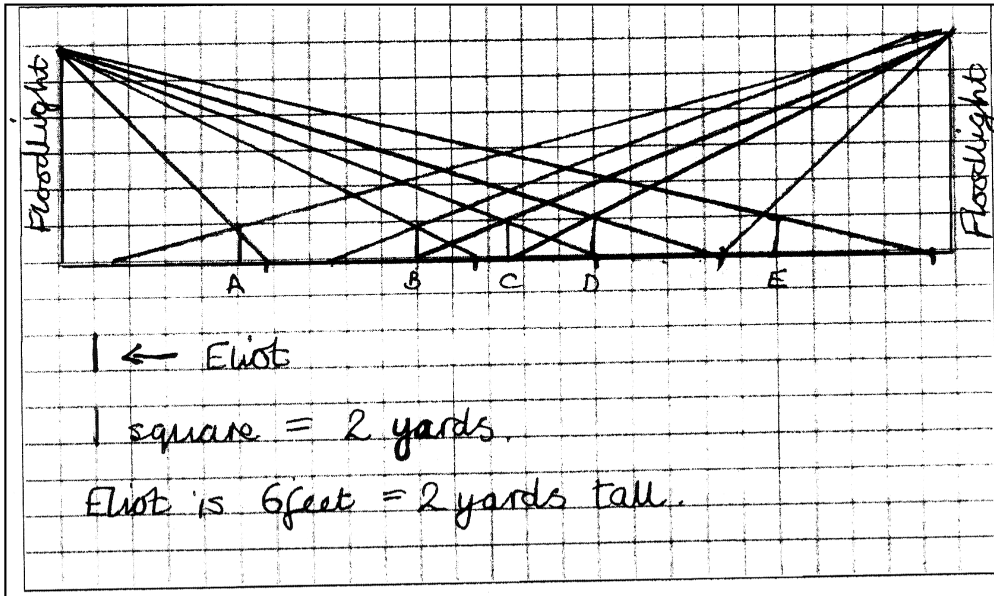
None of the *Sample Responses* are a complete, correct solution.

Work together on one *Sample Response* at a time.

Answer the questions below each piece of work.

- Try to understand what the student has done.
- Explain how the work may be improved.
- Use Tod's and Uma's methods to write finished solutions.

Sample Responses to Discuss: Wendy

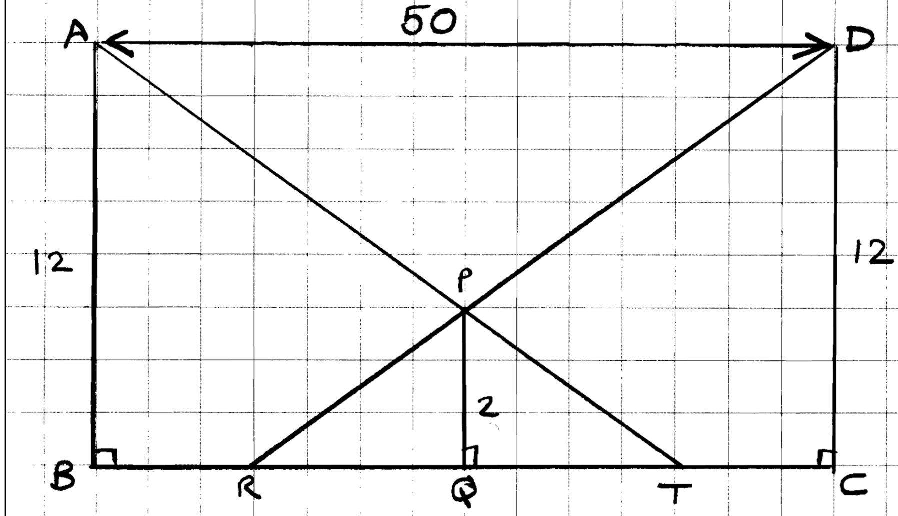


Place Eliot stands	Left shadow	Right shadow	Total shadow
A	$3\frac{1}{2}$	1	$4\frac{1}{2}$
B	$2\frac{1}{2}$	$1\frac{1}{2}$	4
C	$2\frac{1}{2}$	$2\frac{1}{2}$	5
D	2	3	5
E	$1\frac{1}{2}$	$4\frac{1}{2}$	6

As Eliot goes to the right, the left shadow gets shorter and the right shadow gets longer. Total shadow stays about the same.

Sample Responses to Discuss: Tod

Tod



AB and CD are vertical. The football field is horizontal (BC) and so is Eliot. RQ and QT are shadows. I want to find QT.

$QT^2 = PT^2 - 4$ (Pythag).

$PT = AT - AP$

$AP^2 = 10^2 + 25^2$

$AT^2 = 12^2 + (25^2 + QT^2)$

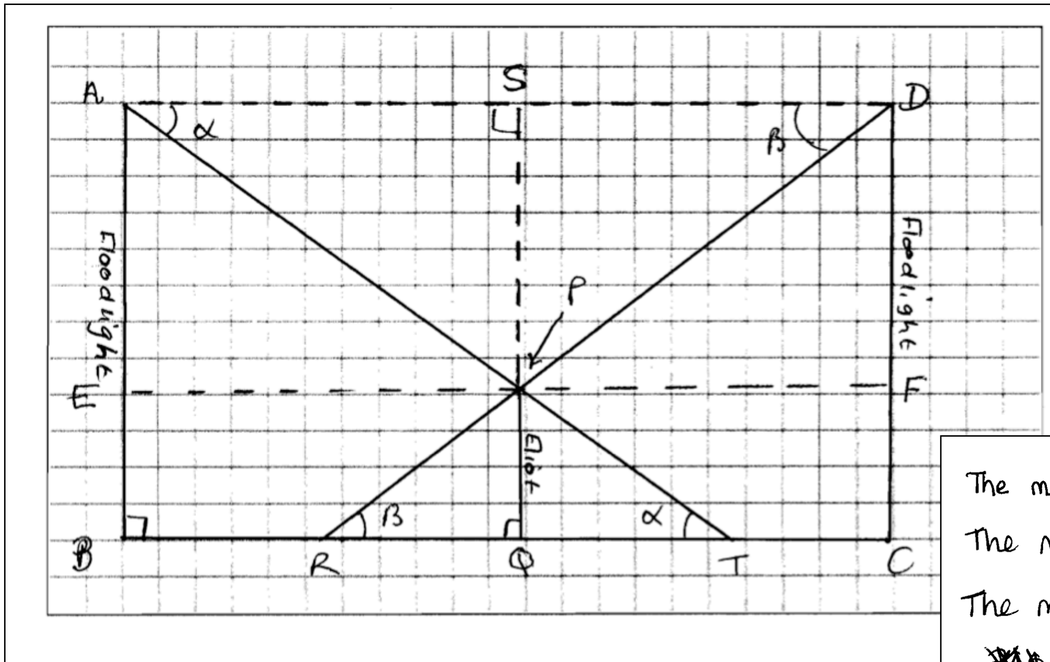
Triangle ABT and triangle PQT are similar so

$$\frac{QT}{PQ} = \frac{BT}{AB}$$

and

$$\frac{QT}{2} = \frac{BT}{12}$$

Sample Responses to Discuss: Uma



The measure of angle TAD = angle PTQ = α.
The measure of angle ADP = angle PRT = β.
The measure of angle APD = angle RPT = $180 - (\alpha + \beta)$.
Triangle RPT is similar to triangle APD.
SP is the perpendicular height of triangle APD.
PQ is the perp. height of triangle PRT.
SP = 12 − 2 = 10 yds PQ = 2 yds.
SP : PQ = 10 : 2 = 5 : 1

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org