CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Classifying Equations of Parallel and Perpendicular Lines

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Classifying Equations of Parallel and Perpendicular Lines

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students understand the relationship between the slopes of parallel and perpendicular lines and in particular, to help identify students who find it difficult to:

- Find, from their equations, lines that are parallel and perpendicular.
- Identify and use intercepts.

It also aims to encourage discussion on some common misconceptions about equations of lines.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

G-PE: Use coordinates to prove simple geometric theorems algebraically.
F-IF: Analyze functions using different representations.

This lesson also relates to **all** the *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 2, 5, and 7:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

The lesson unit is structured in the following way:

- Before the lesson, students work individually on a task designed to reveal their current level of understanding and their difficulties. You then review their work and create questions for students to answer in order to improve their solutions.
- During the lesson, students work in small groups on a related task, justifying and explaining their decisions to each other. They then re-group to critique each other's work. Finally, in a whole-class discussion, students explain and extend their methods and solutions.
- In a follow-up lesson, students work alone to improve their solutions to the assessment task.

MATERIALS REQUIRED

- Each student will need two copies of *Parallel and Perpendicular Lines*, a mini-whiteboard, pen, and eraser.
- Each small group of students will need the *Card Set: Equations* (already cut up into cards), two copies of the *Properties* table, and a glue stick.
- Graph paper should be kept in reserve and only used when requested.
- There is a projector resource to help introduce activities and support the whole-class discussion.

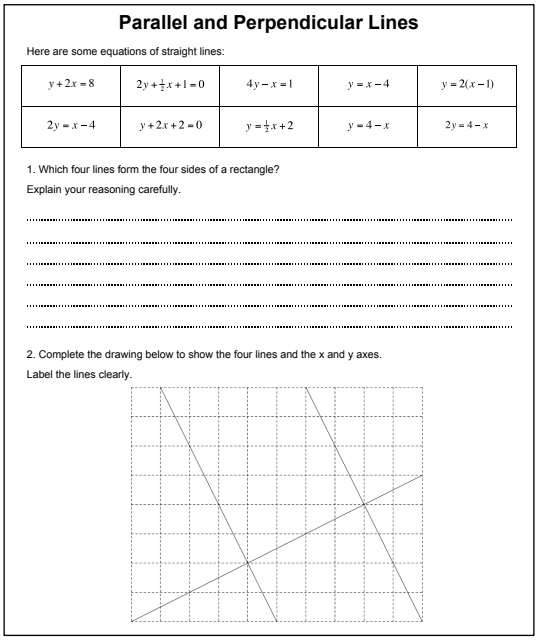
TIME NEEDED

Approximately 15 minutes before the lesson, a 1-hour lesson, and 15 minutes in a follow-up lesson.

BEFORE THE LESSON

Assessment task: *Parallel and Perpendicular Lines* (15 minutes)

Have students do this task, in class or for homework, a day or more before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.



Parallel and Perpendicular Lines

Here are some equations of straight lines:

$y+2x=8$	$2y+\frac{1}{2}x+1=0$	$4y-x=1$	$y=x-4$	$y=2(x-1)$
$2y=x-4$	$y+2x+2=0$	$y=\frac{1}{2}x+2$	$y=4-x$	$2y=4-x$

1. Which four lines form the four sides of a rectangle?
Explain your reasoning carefully.

2. Complete the drawing below to show the four lines and the x and y axes.
Label the lines clearly.

Give each student a copy of the assessment task *Parallel and Perpendicular Lines*.

> *Read through the questions and try to answer them as carefully as you can.*

It is important that, as far as possible, students are allowed to answer the questions without your assistance.

Advise your students that they should not worry too much if they cannot understand or do everything, because there will be a lesson using a similar task that will help them. Explain that their goal is to be able to answer questions such as these by the end of the next lesson.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding. The purpose of doing this is to forewarn you of the difficulties students will experience during the lesson itself, so that you may prepare carefully.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores and will distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this lesson unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students in the follow-up lesson.

Common issues:	Suggested questions and prompts:
Does not link the properties of a rectangle with slopes of lines For example: The student does not mention in Q1 that a rectangle has two pairs of parallel sides and that these pairs are perpendicular, or connect this with the slope of the lines forming the sides.	• What do you know about the sides of rectangles? • How is the property of being parallel [perpendicular] connected with slope? • Why is looking for slopes of pairs of parallel and perpendicular lines relevant?
Demonstrates limited understanding of the link between the slope and the form of the equation of a straight line For example: The student identifies slopes for equations in which y is given explicitly in terms of x, $y = mx + b$, but not for other equations. Or: The student reads the number in front of x as if it were the slope in all equations.	• How do you work out the slope of a line? • How do you work out if two lines are parallel from their slopes? Perpendicular? • Some equations are written with y isolated and others aren't. How does this affect how you calculate the slope?
Provides insufficient reasoning For example: The student does not explain how looking for parallel and perpendicular lines relates to the task of finding the sides of the rectangle (Q1).	• Why is looking for slopes of pairs of parallel and perpendicular lines relevant? • How do you know that...? Explain your reasoning.
Does not identify relevant information from the equations For example: The student fails to identify the x- and/or y-intercepts in Q2.	• What else can you figure out from the equations of the lines? • Where does a pair of lines intersect? • Where does a line intersect the x-axis? • Where does a line intersect the y-axis?

SUGGESTED LESSON OUTLINE

Interactive introduction (10 minutes)

Give each student a mini-whiteboard, pen, and eraser.

Today we're going to be looking at slopes and the equations of parallel and perpendicular lines.

I am going to ask you a series of questions and I would like you to write your answers on your whiteboards.

You may wish to draw a diagram if that helps.

These introductory questions are designed to give you a greater insight into students' misconceptions and are not intended to form a whole-class discussion. The questioning of students should be fast paced and should act as a means of stimulating students to think about the equations of parallel and perpendicular lines. Slides P-1 to P-9 can be used to present these questions if desired.

What is the slope of the line joining the points (2,5) and (7,15)?	*[2.]*
Show me the coordinates of two points for which the slope of the line between them is 3.	*[E.g. (1,4) and (3,10).]*
Give me an equation of a line with a slope of 3.	*[E.g.* $y = 3x + 1$*.]*
Keeping your equation on your whiteboard, write it in a different way.	*[E.g.* $y - 3x = 1$*.]*
What is the y-intercept of the line $y = 3x + 7$*?*	*[(0, 7).]*
What is the x-intercept of the line $2y = 3x - 6$*?*	*[(2, 0).]*
Show me the equation of a line that is parallel to $y = 2x + 4$*.*	*[E.g.* $y = 2x - 6$*.]*
What would be the slope of a line perpendicular to $y = 2x + 4$*?*	*[– ½.]*
Show me the equations of two lines that are perpendicular to each other.	*[E.g.* $y = 2x + 4$ *and* $y = -\frac{1}{2}x + 4$*.]*

During this activity you will be able to identify common misconceptions within the group; this will help you to target groups of students effectively during the collaborative group work.

Collaborative group work: matching task (30 minutes)

Ask students to work in groups of two or three and give each group a copy of the C*ard Set: Equations* (already cut up) and two copies of the *Properties* sheet.

Introduce the activity:

Find two Equations cards to match each of the Properties.

You may want to spend some time thinking about the equations first. It might be helpful to figure out some extra information and write it on the cards.

Once you have found two Equations cards that match a Property, explain to your partner(s) how you came to your decision. If your partner(s) agrees, they should explain your reasoning in their own words. If they disagree, they should explain why they think you are incorrect.

In your group you need to be able to agree on and explain the placement of every card.

Slide P-10 of the projector resource summarizes these instructions.

The purpose of this structured group work is to make students engage with each other's explanations and take responsibility for each other's understanding.

You have two tasks during the small-group work: to note aspects of the task students find difficult and to support students' reasoning.

Note aspects of the task that students find difficult

For example, students may begin by working with the mathematics they understand best and get stuck on later categorizations. You can use information about particular difficulties to focus the whole-class discussion at the end of the lesson.

Support students' reasoning

Try not to make suggestions that move students towards a particular categorization. Instead, ask questions to help students to reason together.

> *How can you determine the slope for any equation? Is there a form of the equation that makes this easier?*
>
> *How do you find the x-intercept from a written equation?*
>
> *If you cannot place all the cards, you may need to rethink some of your categorizations.*

If you find one student has produced a solution for a particular card or match, challenge the other student(s) in the group to provide an explanation.

> *Jenny matched these cards parallel to each other. Jonathon, why does Jenny think these lines are parallel?*

If several students in the class are struggling with the same issue, you could write a relevant question on the board.

A few minutes before the end of the activity, ask one student from each group to write the equations they have succeeded in categorizing onto one copy of the *Properties* sheet.

Once they have done this, ask each small group of students to join with another group, taking with them their *Properties* sheet with equations written on them.

It may be advisable to group students who have displayed a similar level of competence on the task, thus allowing for a richer discussion.

Give each group a glue stick.

> *In your new groups, decide whether or not you agree with each other's answers.*
>
> *If your new partners disagree with your answer, explain your reasoning to them and let them explain why they disagree.*
>
> *Once you are all comfortable with your answers and can explain your reasoning, glue the Equation cards into place on the blank copy of the Properties sheet.*

If students finish early, you may want to ask them to invent an additional equation for each category, using the blank cards. Alternatively, they could create a new heading for the empty box on the *Properties* sheet and write a pair of possible equations for their own choice of category.

Whole-class discussion (20 minutes)

Organize a whole-class discussion. First spend about five minutes considering categorizations and any discrepancies found amongst the groups.

After you changed groups, did your new partners disagree with any of your answers? Give me an example of an equation or pair of equations on which you disagreed. What was the misconception?

Can you explain your reasoning for the final categorization?

Did any other group have different reasoning or categorize those equations under a different heading?

Then spend about 15 minutes generalizing the mathematics students worked on during the lesson.

Display Slide P-11 *Lines and Rectangles*. Tell students that the line segment SP has equation $y = 2x + 3$.

In your groups I want you to decide on a possible equation for each of the line segments PQ, QR, and RS.

Once you are all in agreement, write your equations on your whiteboards and show them to me.

Give students a couple of minutes to come up with a possible solution. Then select an equation from each group and discuss whether or not it is a correct possible solution, focusing on why this is the case.

Follow-up lesson: improve individual responses to the assessment task (15 minutes)

Return students' original assessment tasks, along with a second, blank copy of the task sheet.

If you have not added questions to individual pieces of work, write your list of questions on the board. Students should select from this list only the questions they think are appropriate to their own work.

Look at your original response and the questions (on the board/written on your script.)

Answer these questions and using what you have learned, revise your work.

If students are satisfied with their solutions, ask them to write down a general method for finding the equations of lines that form a rectangle.

Some teachers give this as homework.

SOLUTIONS

Assessment task: *Parallel and Perpendicular Lines*

Question 1. Students may first put the equation into the form $y = mx + b$ and look for m, the slope.

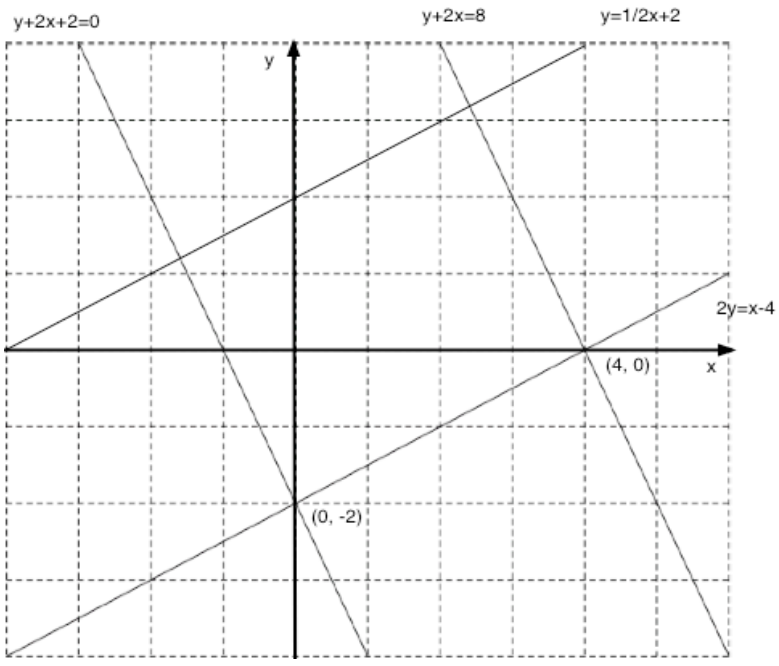
$y + 2x = 8$ $y = -2x + 8$ Slope = –2	$2y + \frac{1}{2}x + 1 = 0$ $y = -\frac{1}{4}x - \frac{1}{2}$ Slope = $-\frac{1}{4}$	$2y + x = 1$ $y = -\frac{1}{2}x + \frac{1}{2}$ Slope = $-\frac{1}{2}$	$y = x - 4$ Slope = 1	$y = 2(x - 1)$ $y = 2x - 2$ Slope = 2
$2y = x - 4$ $y = \frac{1}{2}x - 2$ Slope = $\frac{1}{2}$	$y + 2x + 2 = 0$ $y = -2x - 2$ Slope = –2	$y = \frac{1}{2}x + 2$ Slope = $\frac{1}{2}$	$y = 4 - x$ $y = -x + 4$ Slope = –1	$2y = 4 - x$ $y = -\frac{1}{2}x + 2$ Slope = $-\frac{1}{2}$

The slopes of parallel lines are equal. The product of the slope of a line and its perpendicular is -1.

These pairs of lines are parallel: $y + 2x = 8$ and $y + 2x + 2 = 0$
$2y = x - 4$ and $y = \frac{1}{2}x + 2$
$2y + x = 1$ and $2y = 4 - x$

Lines $y + 2x = 8$ and $y + 2x + 2 = 0$ are perpendicular to $2y = x - 4$ and $y = \frac{1}{2}x + 2$ so these form a rectangle.

Question 2. Lines $y + 2x = 8$ and $y + 2x + 2 = 0$ have a negative slope, so they are the parallel pair shown on the diagram.



Lines $2y = x - 4$ and $y = \frac{1}{2}x + 2$ have a positive slope so either $2y = x - 4$ or $y = \frac{1}{2}x + 2$ is the line that is missing.

The y intercepts of lines $2y = x - 4$ and $y + 2x + 2 = 0$ are the same so these lines cross and intercept the y-axis at the point (0, -2).

Line $y = \frac{1}{2}x + 2$ can be positioned by finding the line that is parallel to $2y = x - 4$ that passes through (0, 2) (y-intercept).

These lines are parallel $2y = 8x + 3$ $y = 4x + 4$	**These lines are perpendicular** $4y = x + 3$ $y + 4x + 6 = 0$
These lines have the same *y*-intercept $y = 6x - 4$ $2y + 8 = 3x$	**These lines have the same *x*-intercept** $3y = 2x - 8$ $2y + x = 4$
These lines go through the point (1,5) $y + 6x = 11$ $y = 8x - 3$	

Parallel and Perpendicular Lines

Here are some equations of straight lines:

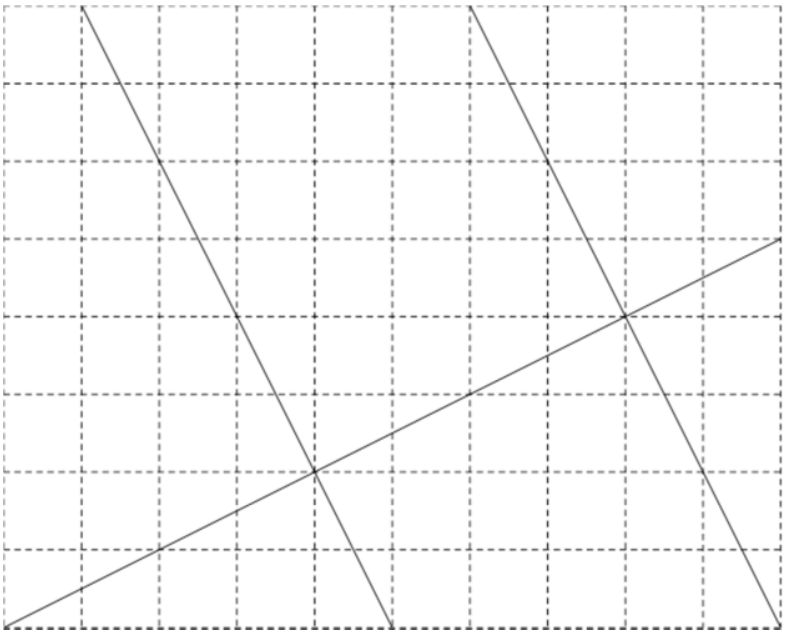
$y+2x=8$	$2y+\frac{1}{2}x+1=0$	$2y+x=1$	$y=x-4$	$y=2(x-1)$
$2y=x-4$	$y+2x+2=0$	$y=\frac{1}{2}x+2$	$y=4-x$	$2y=4-x$

1. Which four lines form the four sides of a rectangle?

   Explain your reasoning carefully.

2. Complete the drawing below to show the four lines and the x- and y-axes.

   Label the lines clearly.



Card Set: Equations

$y = 4x + 4$	$4y = x + 3$
$y = 8x - 3$	$y + 4x + 6 = 0$
$3y = 2x - 8$	$y + 6x = 11$
$2y + 8 = 3x$	$2y + x = 4$
$2y = 8x + 3$	$y = 6x - 4$

Properties

These lines are parallel	These lines are perpendicular
These lines have the same y-intercept	**These lines have the same x-intercept**
These lines go through the point (1,5)	

Question

What is the slope of the line joining the points (2,5) and (7,15)?

Question

Show me the coordinates of two points for which the slope of the line between them is 3.

Question

Give me an equation of a line with a slope of 3.

Question

Keeping your equation on your whiteboard, write it in a different way.

Question

What is the y-intercept of the line $y = 3x + 7$?

Question

What is the x-intercept of the line $2y = 3x - 6$?

Question

Show me the equation of a line that is parallel to $y = 2x + 4$.

Question

What would be the slope of a line perpendicular to $y = 2x + 4$?

Question

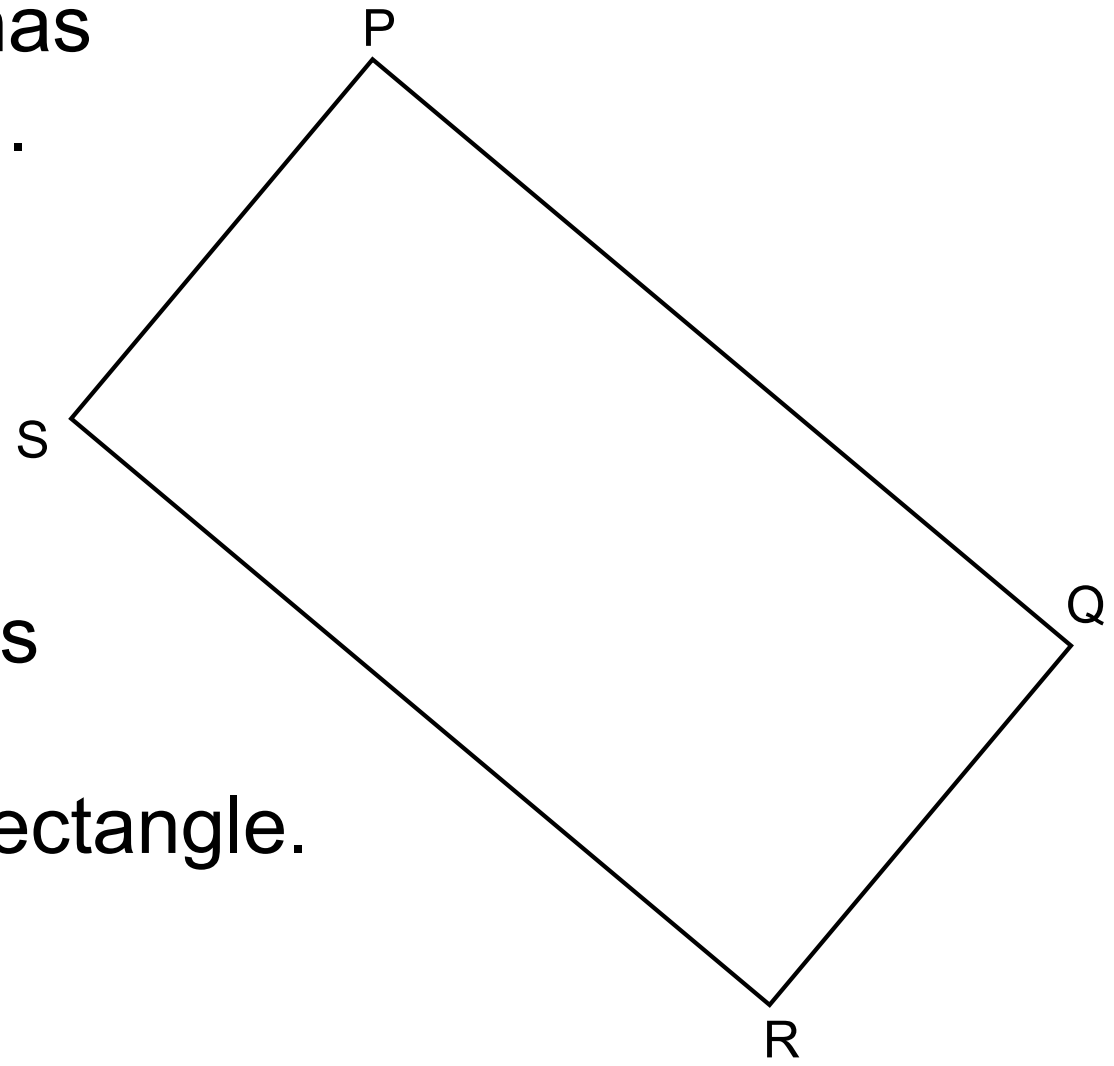
Show me the equations of two lines that are perpendicular to each other.

Collaborative group work

- Find two *Equations* cards to match each *Property*.
- You may want to think about the equations first. You can write extra information on the cards.
- Once you have found two *Equations* cards that match a *Property*, explain to your partner(s) how you came to your decision.
- If your partner(s) agrees, they should explain your reasoning in their own words. If they disagree, they should explain why they think you are incorrect.
- In your group you need to be able to agree on and explain the placement of every card.

Lines and Rectangles

Line segment SP has equation $y = 2x + 3$.



Find the equations of the line segments forming the other three sides of the rectangle.

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org