CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Building and Solving Complex Equations

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Building and Solving Complex Equations

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to create and solve linear and non-linear equations. In particular, the lesson will help identify and help students who have the following difficulties:

- Solving equations where the unknown appears once or more than once.
- Solving equations in more than one way.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

A-REI: Understand solving equations as a process of reasoning and explain the reasoning.
Solve equations and inequalities in one variable.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 1, 7, and 8.

1. Make sense of problems and persevere in solving them.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

The unit is structured in the following way:

- Before the lesson, students work individually on an assessment task that is designed to reveal their current levels of understanding and difficulties. You then review their work and create questions for students to consider when improving their solutions.
- Students work collaboratively building and solving equations in which the unknown appears more than once in the equation.
- During a final whole-class discussion students review the main mathematical concepts of the lesson.
- In a follow-up lesson, students review their initial solutions and then use what they have learned to either revise the same introductory assessment task or complete a different task.

MATERIALS REQUIRED

- Each student will need a copy of the assessment tasks, *Building and Solving Equations* and *Building and Solving Equations (revisited),* the cut-up sheet *Building Equations*, the cut-up sheet *Solving Equations*, a sheet of paper, a mini-whiteboard, a pen, and an eraser. Some students may need extra copies of the sheets *Building Equations* and *Solving Equations*.
- There is a projector resource to support whole-class discussions.

TIME NEEDED

15 minutes before the lesson, a 70-minute lesson (or two shorter lessons), and 15 minutes in a follow-up lesson. Exact timings will depend on the needs of your class.

BEFORE THE LESSON

Assessment task: *Building and Solving Equations* (15 minutes)

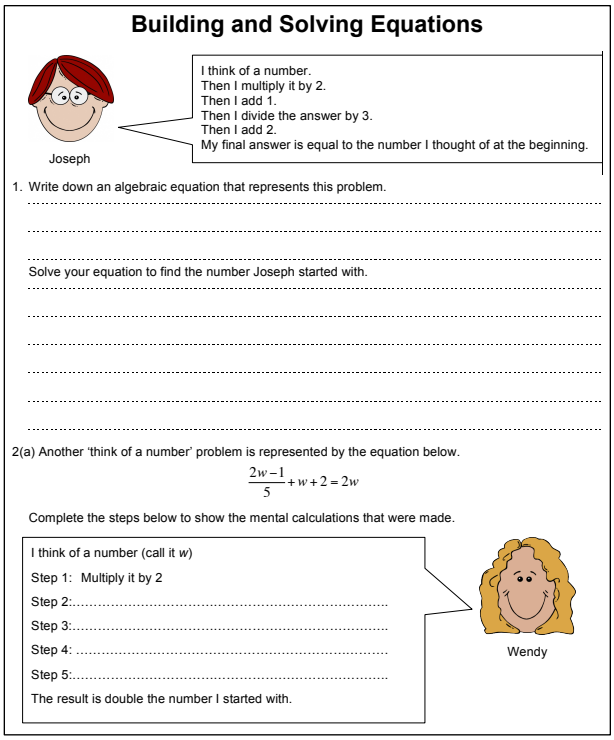
Have students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the next lesson.

Give each student a copy of the assessment task *Building and Solving Equations*.

> *Read through the questions and try to answer them as carefully as you can.*

It is important that students are allowed to answer the questions without your assistance, as far as possible.

Students should not worry too much if they cannot understand or do everything, as in the next lesson, they will engage in a similar task that should help them to progress. Explain to students that by the end of the next lesson, they should expect to answer questions such as these confidently. This is their goal.



Building and Solving Equations

Joseph

I think of a number.
Then I multiply it by 2.
Then I add 1.
Then I divide the answer by 3.
Then I add 2.
My final answer is equal to the number I thought of at the beginning.

1. Write down an algebraic equation that represents this problem.

Solve your equation to find the number Joseph started with.

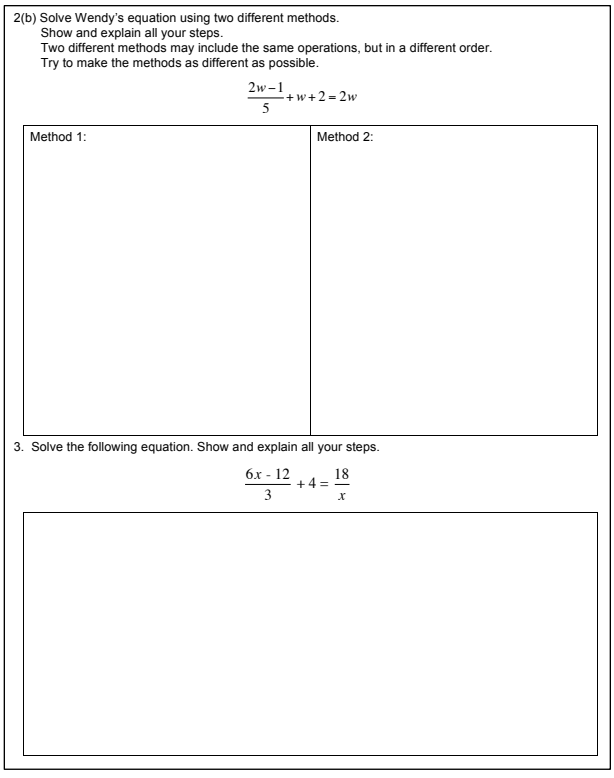
2(a) Another 'think of a number' problem is represented by the equation below.

$$\frac{2w-1}{5} + w + 2 = 2w$$

Complete the steps below to show the mental calculations that were made.

I think of a number (call it *w*)
Step 1: Multiply it by 2
Step 2:.....
Step 3:.....
Step 4: .....
Step 5:.....
The result is double the number I started with.

Wendy



2(b) Solve Wendy's equation using two different methods.
Show and explain all your steps.
Two different methods may include the same operations, but in a different order.
Try to make the methods as different as possible.

$$\frac{2w-1}{5} + w + 2 = 2w$$

Method 1:	Method 2:

3. Solve the following equation. Show and explain all your steps.

$$\frac{6x - 12}{3} + 4 = \frac{18}{x}$$

Assessing students' responses

Collect students' responses to the task and note what their work reveals about their current levels of understanding and their individual difficulties.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores and will distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a list of questions. Some suggestions for these are given in the *Common issues* table on the next page. We suggest that you make a list of your own questions, based on your students' work, using the ideas on the following page. We recommend you:

- write one or two questions on each student's work, or

- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these questions on the board when you return the work to the students in the follow-up lesson.

Common issues:	**Suggested questions and prompts:**
When solving the equation, different operations are applied to its two sides For example (Q1): The student writes: $2x + 1 = 3x - 6$; $2x - 5 = 3x$ instead of $2x + 7 = 3x$.	• How can you check your solution is correct? • Substitute your final answer into each side of the equation. What do you get?
When solving or making an equation, there are incorrect additions or subtractions of a value to or from an expression For example: The student attempts to add 2 to both sides of the equation (Q1): Add 2 $\frac{2x+1}{3} = x - 2$ $\Rightarrow$ $\frac{2x+3}{3} = x$	• Write the expression with the one fraction bar as two fractions. What do you get when you add 2? Write the resultant expression on the left side of the equation as one fraction. $\frac{2x}{3} + \frac{1}{3} = x - 2$ • After the addition, are the two sides of the equation still equal? How do you know?
When building or solving an equation, incorrect multiplication of just one term in the expression For example (Q1): Multiplying $x - 2$ by 3, the student writes $3x - 2$ instead of $3(x - 2)$, or $3x - 6$.	• Multiply by 3 means multiply the whole expression by 3. How could you write this? • Think of words to describe the expression with parentheses. Do these words match your expression without the parentheses?
When solving an equation, incorrect multiplication of an expression For example: The student may decide to multiply both sides of the equation by 5 (Q2): Multiply by 5 $\frac{2w-1}{5} + w + 2 = 2w$ results in $2w - 1 + w + 2 = 10w$ instead of $2w - 1 + 5w + 10 = 10w$.	• Are both sides of your equation equal? How do you know? • Have you multiplied all terms by 5? • Now divide both sides of the equation by 5. Does this give you your original equation?
Only one answer is provided For example (Q3): The student writes $x = 3$ for the equation $x^2 = 9$.	• Is the equation linear or quadratic? What do you know about these types of equations?
All the answers are correct	• Now use a different method to solve equation 3.

SUGGESTED LESSON OUTLINE

In this lesson students build and solve complex-looking equations. Understanding how to construct these equations will help students to solve them using a similar step-by-step 'de-construction' approach.

If you think your students will struggle with the introduction to the lesson you may want to begin by using the first part of the Middle School lesson *Building and Solving Linear Equations*.

Whole-class introduction (20 minutes)

Throughout this discussion there are some challenging problems; encourage students to first tackle these individually and only then discuss it with a neighbor. In that way students will have something to talk about and may help prevent a few students dominating the discussion. Maximize participation by asking all students to show you solutions on their mini-whiteboards. Select a few students with interesting or contrasting answers to justify them to the class. Encourage the rest of the class to contest these explanations.

Depending on how the discussion goes, you may not need to use all the examples provided here.

Give each student a mini-whiteboard, pen, and eraser.

In this lesson you will build equations from solutions such as $x = 6$ and solve equations.

The building of equations will help you understand how to solve them, that is 'un-build' them.

Write on the board or an overhead transparency:

$$x = 6$$

Then state that this value for x could have been derived from the equation:

$$3x = 2x + 6$$

Is there only one solution to this equation? [Yes.]

Is there only one equation that will give this solution? [No.]

Ask students to construct an equation where x appears on both sides, that would solve to give the answer $x = 6$.

Encourage students to use more than one operation; for example students may consider using subtraction, multiplication, division, and equations with x^2.

Show me a more complicated equation where the solution is still $x = 6$.

Show me an equation with a division, where the solution is still $x = 6$.

Show me an equation that uses parentheses, where the solution is still $x = 6$.

Show me an equation with a division by x, where the solution is still $x = 6$.

After a few minutes ask students to show you their mini-whiteboards.

Here are some possible examples:

$4x = 3x + 6$ or $2x + 3 = 9 + x$

or $3x - 6 = 2x$ or $4x^2 = (6 + x)^2$

or $\frac{x}{2} = 6 - \frac{x}{2}$ or $\frac{3x}{2} - 1 = 2 + x$

or $x = \frac{36}{x}$ or $\frac{3x - 2}{4} = x - 2$

Ask two or three students with quite different equations to explain how they arrived at them.

Could your method be used to build any equation?

Has anyone come up with a different way to create an equation?

How could we check that these equations are correct? [Solve them or substitute $x = 6$ into them.]

Which is the easier method? Why? [Substitution is easier with the more difficult equations.]

Do any of the equations have more than one solution for x? Which ones? Please explain.

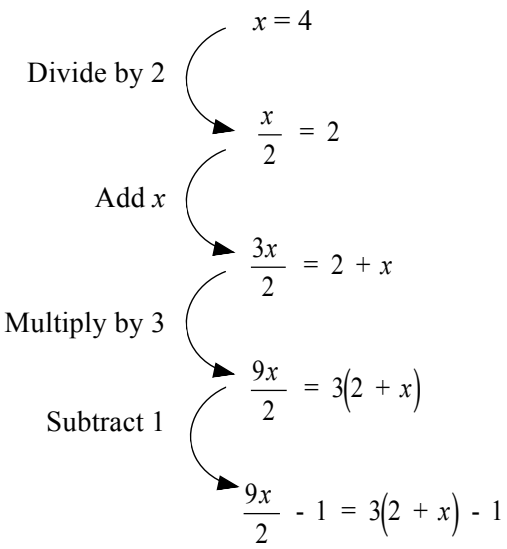
There are many methods students may use to create equations. Some use guess and check; some build up an equation incrementally by performing the same operation on both sides of the equation.

In this lesson we will build equations in a step-by-step manner.

At each step, the same operation will be applied to each side of the equation.

In this way we can create complex-looking equations!

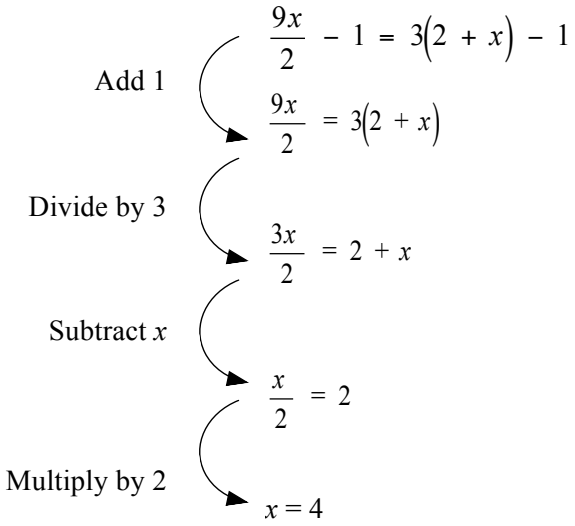
Write $x = 4$ on the board and ask students to suggest operations to build up an equation in a step-by-step manner. One of the operations should involve x. For example:



Substitute the original value of x to check that the equation is correct.

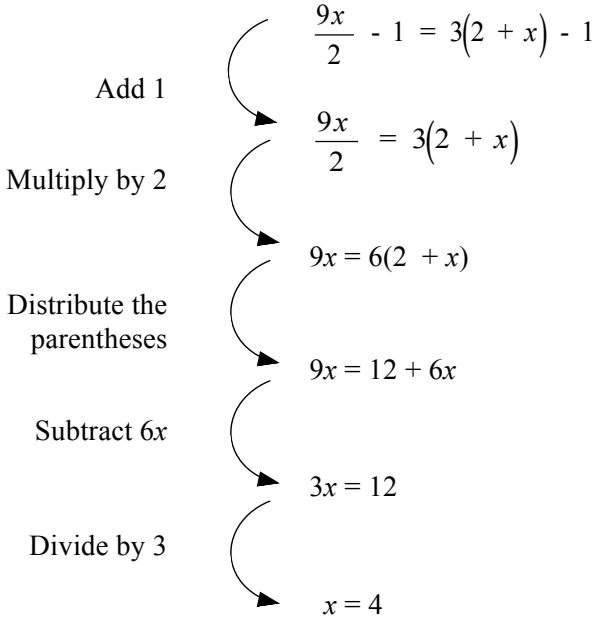
Suppose now that you were given this final equation. How would you solve it?

Ask students to demonstrate how the resulting equation may be solved by reversing the sequence of operations and carrying out the inverse of each one:



The equation may be solved by reversing the operations that were used to build it, but there are also other ways to solve the equation.

Discuss how the example may be solved in other ways. For example:



Collaborative activity (30 minutes)

Give each student the cut-up sheets *Building Equations* and *Solving Equations*.

You are going to build two equations for your partner to solve.
I want you to do this individually.

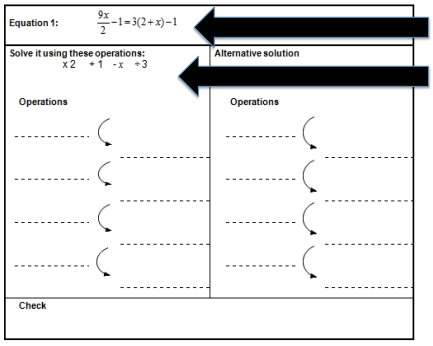
Explain the first task: *Building and Checking an Equation*, using Slide P-1:

Building and Checking an Equation

1. Make up your own value for x.
2. Build an equation. Use each of the four operations +, −, ×, and ÷ and different integers.
3. Make sure x appears on both sides of the final equation.
4. Use substitution to check that your equation works.
5. Now make up a second equation.

When students have created two equations, ask them to write the final equation on the *Solving Equations* sheet. In the first column, they should also try to provide help, describing the operations needed to solve the equation. This is explained on Slide P-2:

Using the Sheet: Solving Equations



Write finished equation here.

Write operations need to solve it here, in any order.

For example, if you added $2x$ to both sides, write $-2x$. If you divided both sides by 3 then write x3.

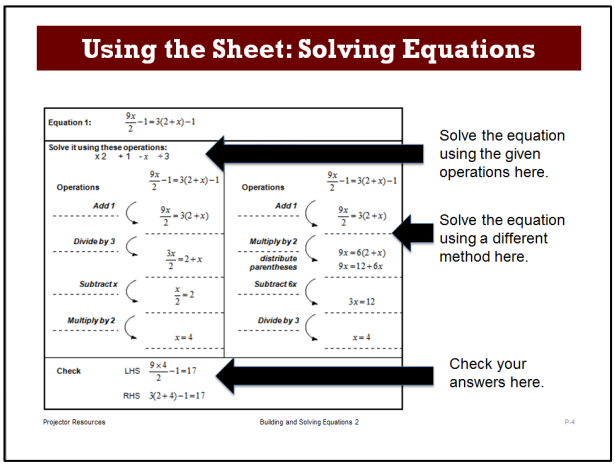
Now hand the sheet to your partner.

The sheet should be handed to the partner, who must try to solve each equation in two different ways. Explain the second task: *Solving Equations*, using Slides P-3 and P-4:

Working Together: Solving Equations

1. Ask your partner to solve each equation.
   - Solve one equation using the operations provided;
   - Solve the same equation using a different method.
2. Help your partner if they become stuck.
3. If your partner's answers are different from yours, ask for an explanation. If you still don't agree, explain your own thinking.

It is important that you both agree on the answers.



The purpose of this structured group work is to make students engage with each other's explanations and take responsibility for each other's understanding.

While students are working in small groups you have two tasks: to note how students approach the task and to support student reasoning.

Note different student approaches to the task

Notice how students make a start on the task, where they get stuck, and how they respond if they do come to a halt. Notice any errors. Students may make calculation errors when substituting into equations. They may use non-standard notation. Students may not perform operations on both sides of the equation, or not use the distributive property accurately when solving equations. Students may multiply or divide just one term in the expression, instead of the whole expression on both sides of the equation. When solving an equation, students may not realize there could be more than one solution.

You can use this information to focus the whole-class discussion at the end of the lesson.

Support student reasoning

Try not to make suggestions that resolve errors and difficulties for students. Instead, ask questions to help students to reason together to identify and resolve issues.

> *If the unknown is on both sides of the equation, how can you eliminate it from one side?*
>
> *Explain how you know which operation to undo first.*
>
> *How can you now check that you are correct?*
>
> *Can you find a different way of writing this expression?*
>
> *How do you know these two expressions are equal?*

Comparing building and solving equations:

> *Can you use all the operations provided by your partner? Why/Why not?*
> *[The partner, when building the equation may have simplified an expression.]*
>
> *For the first solution, are the operations you used in the exact reverse order to that used to build it? If not, why not?*
>
> *Can you use the operations in a different order and still get the correct answer? Why/Why not?*
>
> *Which method do you prefer? Why?*

The questions in the *Common issues* table may also help you support your students.

Encourage students who quickly complete the two sheets to create more challenging equations without using the structured sheets. For example, they may want to include x^2 or $1/x$ in their equations.

Whole-class discussion (20 minutes)

Organize a discussion about what has been learned. Depending on how the lesson went, you may want to focus on the common mistakes students made, review what has been learnt, or extend and generalize the math.

Throughout this discussion encourage students to justify their answers. Try not to correct answers, but encourage students to challenge each other's explanations.

Write this equation on the board:

$$\frac{5\left(\frac{b-1}{2}-1\right)+2b}{3}=5+b$$

Show me a method for solving this equation.

After a few minutes ask students to show you their whiteboards. Ask two or three students with different answers to justify them to the rest of the class.

This is one possible method:

$$\frac{5\left(\frac{b-1}{2}-1\right)+2b}{3}=5+b$$

Multiply by 3

$$5\left(\frac{b-1}{2}-1\right)+2b=15+3b$$

Distribute the 5

$$\frac{5b-5}{2}-5+2b=15+3b$$

Add 5

$$\frac{5b-5}{2}+2b=20+3b$$

Subtract $3b$

$$\frac{5b-5}{2}-b=20$$

Multiply by 2

$$5b-5-2b=40$$

Simplify

$$3b-5=40$$

Add 5

$$3b=45$$

Divide by 3

$$b=15$$

Ask students to critique each other's solution methods.

Does anyone disagree with this method?

Does anyone have a different method?

Does anyone have a more efficient method?

Follow-up lesson: reviewing the assessment task (15 minutes)

Return to the students their original assessment *Building and Solving Equations*.

If you have not added questions to individual pieces of work, or highlighted questions on a sheet of paper, then write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

Read through your original solutions to the task and think about what you have learned this lesson.

Carefully read through the questions I have written.

Use what you have learnt to answer the questions.

If students struggled with the original assessment then they may benefit from revising this assessment. In order that students can see their own progress, ask them to complete the task using a different color pen or give them a second blank copy of the task. Otherwise give students a copy of the task *Building and Solving Equations (revisited)*.

Use what you have learned to complete the new assessment task/revise your answers.

You could give this task for homework.

SOLUTIONS

Assessment task: *Building and Solving Equations*

There are several ways to solve each equation. Below are some examples.

1. $\frac{2x+1}{3}+2=x$

Subtract 2	$\frac{2x+1}{3}=x-2$
Multiply by 3	$2x+1=3(x-2)$
Distribute parentheses	$2x+1=3x-6$
Add 6	$2x+7=3x$
Subtract $2x$	$7=x$

2. I think of a number (call it w)

Step 1: Multiply it by 2

Step 2: Subtract 1

Step 3: Divide by 5

Step 4: Add the number I started with

Step 5: Add 2

The result is double the number I started with.

Solving the equation:

$$\frac{2w - 1}{5} + w + 2 = 2w$$

Subtract w	$\frac{2w - 1}{5} + 2 = w$
Multiply by 5	$2w - 1 + 10 = 5w$
Simplify	$2w + 9 = 5w$
Subtract $2w$	$9 = 3w$
Divide by 3	$3 = w.$

3.

	$\frac{6x - 12}{3} + 4 = \frac{18}{x}$
Multiply by 3	$6x - 12 + 12 = \frac{54}{x}$
Simplify	$6x = \frac{54}{x}$
Multiply by x	$6x^2 = 54$
Divide by 6	$x^2 = 9$
Square root	$x = 3$ or $x = -3$

Assessment task: *Building and Solving Equations (revisited)*

There are several ways to solve each equation. Below are some examples.

1. $\frac{3y+4}{4} = y-3$

Multiply by 4	$3y+4 = 4(y-3)$
Distribute parentheses	$3y+4 = 4y-12$
Add 12	$3y+16 = 4y$
Subtract $3y$	$16 = y$

2. I think of a number (call it w)

Step 1: Multiply it by 3

Step 2: Add 5

Step 3: Divide by 4

Step 4: Add 2

Step 5: Multiply by 4

Step 6: Subtract 3

The result is five times the number I started with.

Solving the equation:

$$4\left(\frac{3w+5}{4}+2\right) - 3 = 5w$$

Distribute	$3w+5+8-3 = 5w$
Simplify	$3w + 10 = 5w$
Subtract $2w$	$10 = 2w$
Divide by 2	$5 = w$.

3. $\frac{24+2x}{x} - 1 = \frac{2x+3}{3}$

Multiply by 3	$\frac{72+6x}{x} - 3 = 2x+3$
Multiply by x	$72+6x-3x = 2x^2+3x$
Simplify	$72+3x = 2x^2+3x$
Subtract $3x$	$72 = 2x^2$
Divide by 2	$36 = x^2$
Square root	$x = +6$ or $x = -6$

Building and Solving Equations

Joseph

I think of a number.
Then I multiply it by 2.
Then I add 1.
Then I divide the answer by 3.
Then I add 2.
My final answer is equal to the number I thought of at the beginning.

1. Write down an algebraic equation that represents this problem.

Solve your equation to find the number Joseph started with.

2(a) Another 'think of a number' problem is represented by the equation below.

$$\frac{2w-1}{5}+w+2=2w$$

Complete the steps below to show the mental calculations that were made.

I think of a number (call it *w*)

Step 1: Multiply it by 2

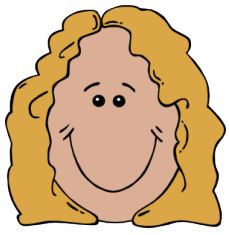
Step 2:……

Step 3:……

Step 4: ……

Step 5:……

The result is double the number I started with.



Wendy

2(b) Solve Wendy's equation using two different methods.
Show and explain all your steps.
Two different methods may include the same operations, but in a different order.
Try to make the methods as different as possible.

$$\frac{2w-1}{5}+w+2=2w$$

Method 1:	Method 2:

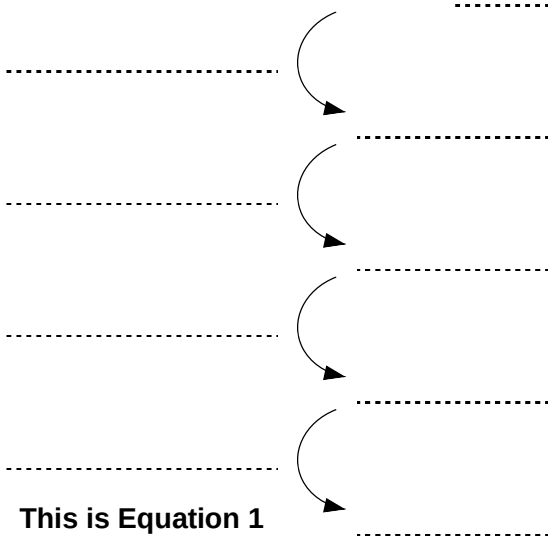
3. Solve the following equation. Show and explain all your steps.

$$\frac{6x - 12}{3} + 4 = \frac{18}{x}$$

Building Equations

Operations

x =

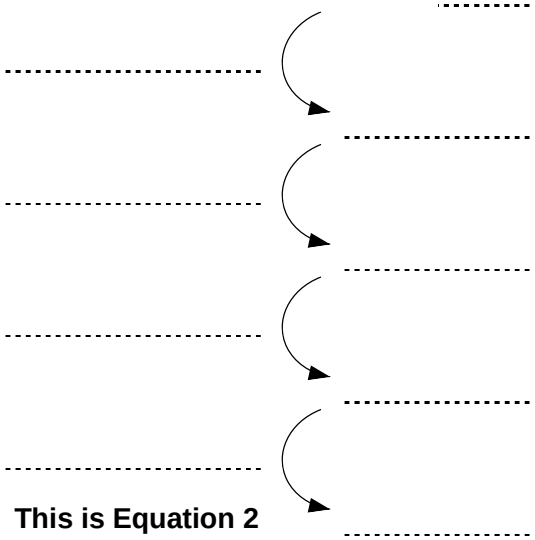


This is Equation 1

Check

Operations

y =

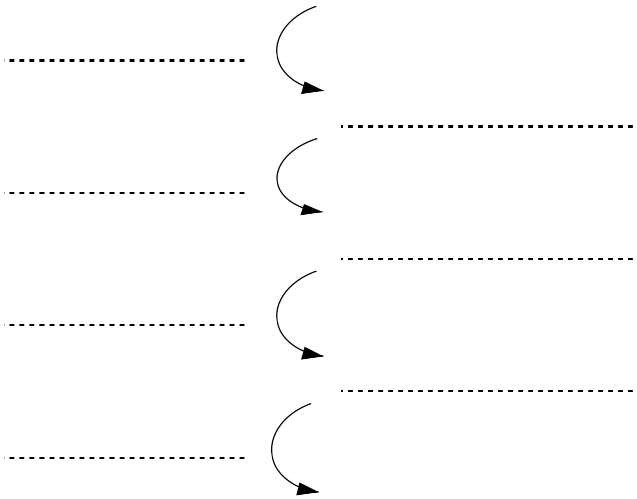
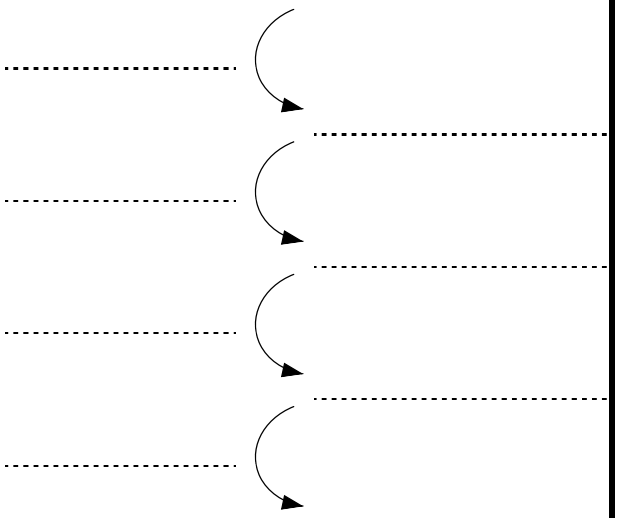


This is Equation 2

Check

Solving Equations

Equation 1:

Solve it using these operations:	**Alternative solution**
Operations	**Operations**
	
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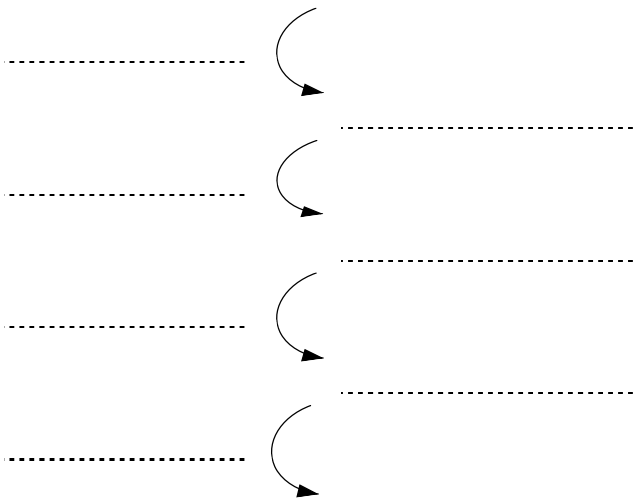
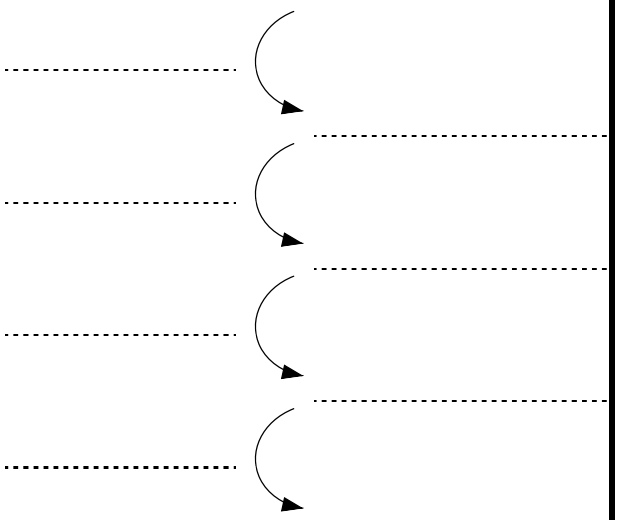
Equation 2:

Solve it using these operations:	**Alternative solution**
Operations	**Operations**
	
Check	

Building and Solving Equations (revisited)

Max

I think of a number.
Then I multiply it by 3.
Then I add 4.
Then I divide the answer by 4.
My final answer is 3 less than the number I thought of at the beginning.

1. Write down an algebraic equation that represents this problem.

Solve your equation to find the number Max started with.

2(a) Another 'think of a number' problem is represented by the equation below.

$$4\left(\frac{3w+5}{4}+2\right)-3=5w$$

Complete the steps below to show the mental calculations that were made.

I think of a number (call it *w*)

Step 1: Multiply it by 3

Step 2:……..

Step 3:……..

Step 4: ……

Step 5:……..

Step 6: ……

The result is five times the number I started with.

Colleen

2(b) Solve Colleen's equation using two different methods.
Show and explain all your steps.
Two different methods may include the same operations, but in a different order.
Try to make the methods as different as possible.

$$4\left(\frac{3w+5}{4}+2\right)-3=5w$$

Method 1:	Method 2

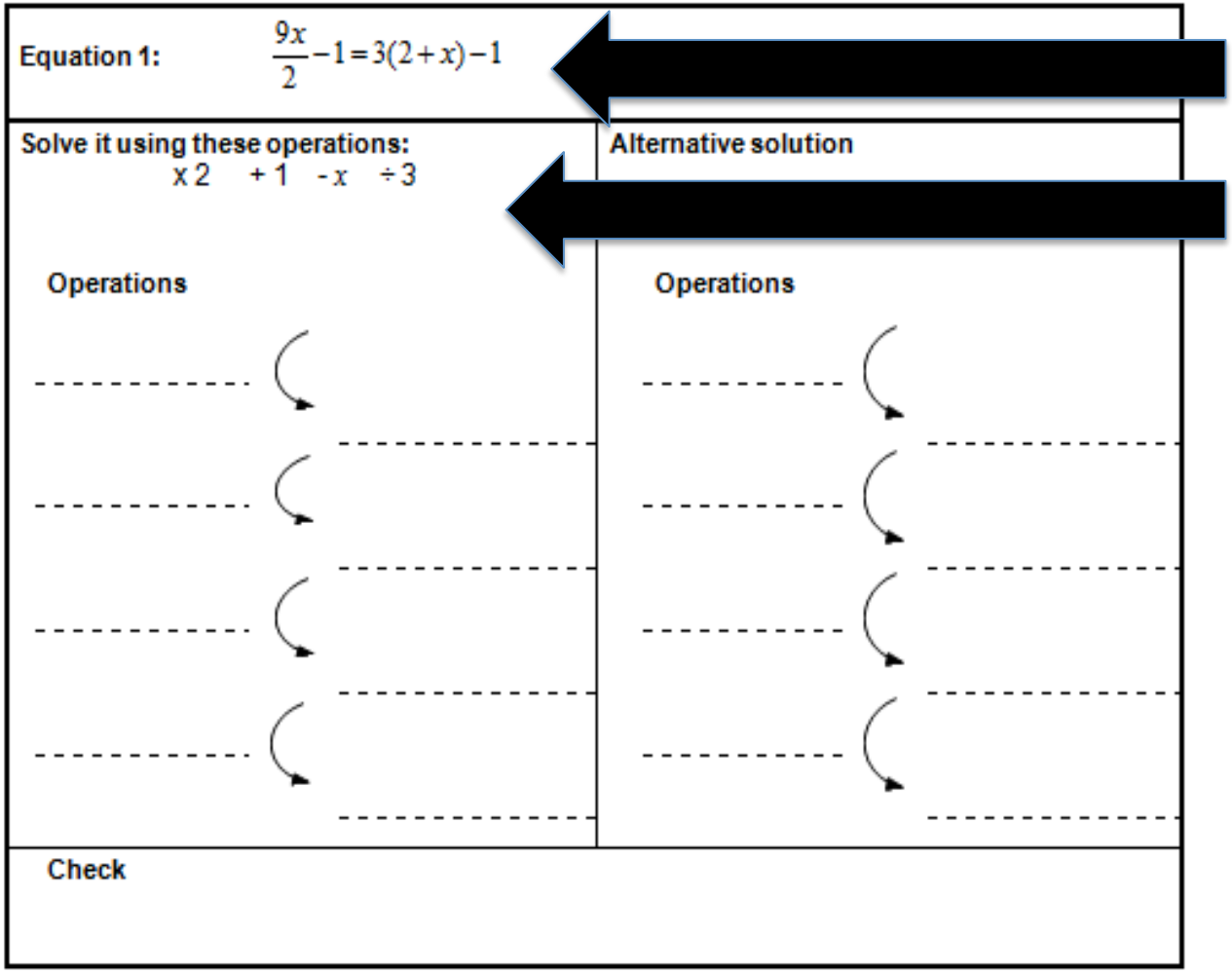
3. Solve the following equation. Show and explain all your steps.

$$\frac{24 + 2x}{x} - 1 = \frac{2x + 3}{3}$$

Building and Checking an Equation

1. Make up your own value for x.
2. Build an equation. Use each of the four operations +, −, ×, and ÷ and different integers.
3. Make sure x appears on both sides of the final equation.
4. Use substitution to check that your equation works.
5. Now make up a second equation.

Using the Sheet: Solving Equations



Write finished equation here.

Write operations need to solve it here, in any order.

For example, if you added $2x$ to both sides, write $-2x$.
If you divided both sides by 3 then write x3.

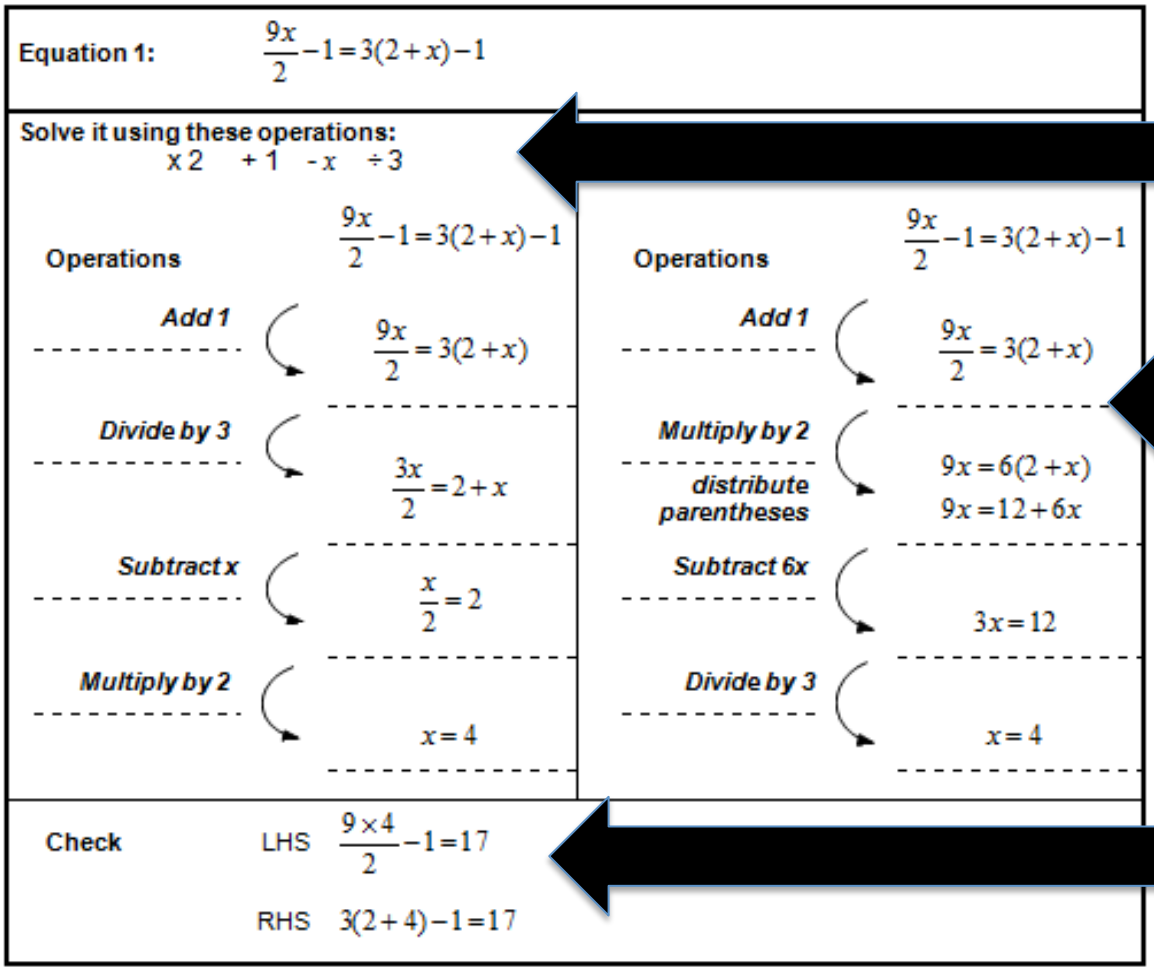
Now hand the sheet to your partner.

Working Together: Solving Equations

1. Ask your partner to solve each equation.
   - Solve one equation using the operations provided;
   - Solve the same equation using a different method.
2. Help your partner if they become stuck.
3. If your partner's answers are different from yours, ask for an explanation. If you still don't agree, explain your own thinking.

It is important that you both agree on the answers.

Using the Sheet: Solving Equations



Solve the equation using the given operations here.

Solve the equation using a different method here.

Check your answers here.

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
Shell Center Team at the Center for Research in Mathematical Education
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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org