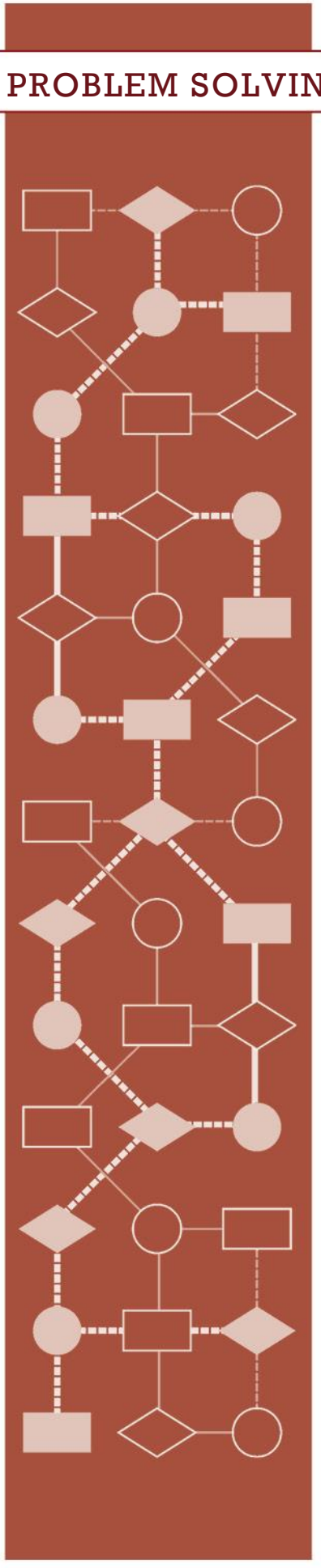
PROBLEM SOLVING



Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Designing: *A Game of Chance*

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Designing: *A Game of Chance*

MATHEMATICAL GOALS

This lesson unit is intended to help students to:

- Understand the connection between relative frequency and probability.
- Gain experience of designing and carrying out a probability experiment.
- Calculate a theoretical probability for a practical scenario.

COMMON CORE STATE STANDARDS

This lesson relates to **all** the *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 2, 3, 4, and 5:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

7.SP: Investigate chance processes and develop, use, and evaluate probability models.

7.G: Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

INTRODUCTION

The lesson unit is structured in the following way:

- Before the lesson, students attempt the *Throwing a Coin* task individually. You then review their responses and create questions for students to consider to help them to improve their work.
- At the start of the lesson, students think individually about their responses to the questions set.
- Next, students work in pairs or threes, combining their thinking and working together to decide which game design will make the most money for charity. In the same small groups, students evaluate sample responses and compare each approach with their own ideas. They then review their work to ensure they are happy with and can justify their solution.
- In a final whole-class discussion, students share their work and consider the advantages and disadvantages of the approaches taken.
- In a follow-up lesson, students reflect on how they have worked.

MATERIALS REQUIRED

- Each student will need the assessment task *Throwing a Coin*, some plain paper to work on, a mini-whiteboard, pen and eraser, and the *How Did You Work?* questionnaire. Calculators and rules should be made available on request.
- Each small group of students will need plain paper to work, a copy of the *Game Board*, a quarter, a 50-cent piece, and the *Sample Responses to Discuss*.

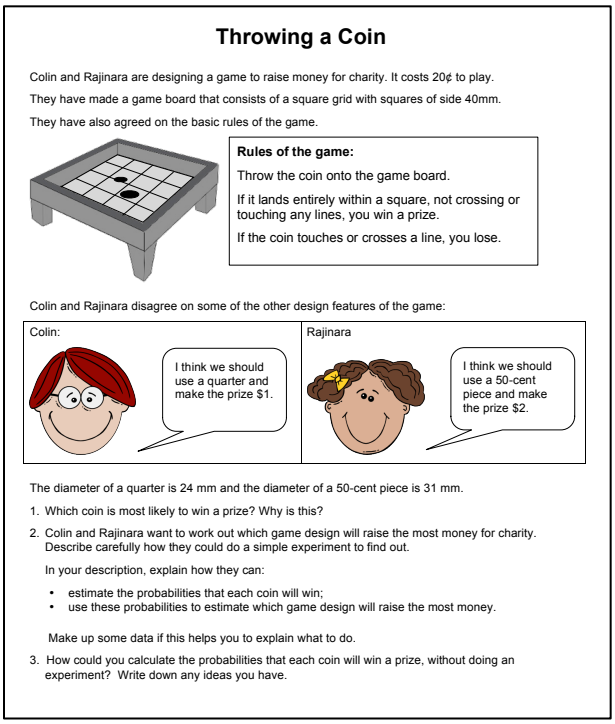
TIME NEEDED

15 minutes before the lesson, a 110-minute lesson (or two shorter lessons), and 15 minutes in a follow-up lesson. Timings are approximate and will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *Throwing a Coin* (15 minutes)

Have the students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.



Throwing a Coin

Colin and Rajinara are designing a game to raise money for charity. It costs 20¢ to play.

They have made a game board that consists of a square grid with squares of side 40mm.

They have also agreed on the basic rules of the game.

Rules of the game:

Throw the coin onto the game board.

If it lands entirely within a square, not crossing or touching any lines, you win a prize.

If the coin touches or crosses a line, you lose.

Colin and Rajinara disagree on some of the other design features of the game:

Colin: I think we should use a quarter and make the prize $1.

Rajinara: I think we should use a 50-cent piece and make the prize $2.

The diameter of a quarter is 24 mm and the diameter of a 50-cent piece is 31 mm.

1. Which coin is most likely to win a prize? Why is this?
2. Colin and Rajinara want to work out which game design will raise the most money for charity. Describe carefully how they could do a simple experiment to find out.

   In your description, explain how they can:

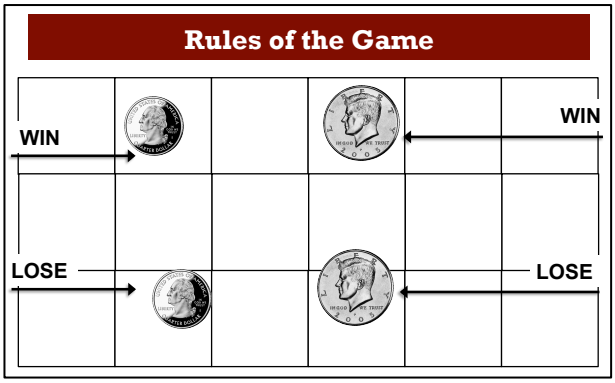
   - estimate the probabilities that each coin will win;
   - use these probabilities to estimate which game design will raise the most money.

   Make up some data if this helps you to explain what to do.
3. How could you calculate the probabilities that each coin will win a prize, without doing an experiment? Write down any ideas you have.

Give each student a copy of *Throwing a Coin* and some plain paper to work on. Introduce the task:

> *Colin and Rajinara are designing a game to raise money for charity and can't agree on some of the features of the game.*
>
> *They have already designed the game board and agreed on the basic rules of the game.*

Display Slide P-1 of the projector resource and check that students understand the rules of the game:



Explain to students that to win, the coin (whether they agree upon a quarter or a 50-cent piece) has to land entirely within a square. If it crosses a line or touches a line, the player loses. Once students are happy with this, ask them to complete the questions:

> *Read the sheet carefully and answer the questions, explaining your work as carefully as you can.*

It is important that, as far as possible, students are allowed to work on the task without your assistance.

Students should not worry too much if they cannot understand or do everything, because in the next lesson they will engage in a similar task, which should help them. Explain to students that by the end of the next lesson, they should expect to be able to answer questions like these confidently. This is their goal.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem-solving approaches.

We suggest that you do not score students' work. Research suggests that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students at the beginning of the lesson.

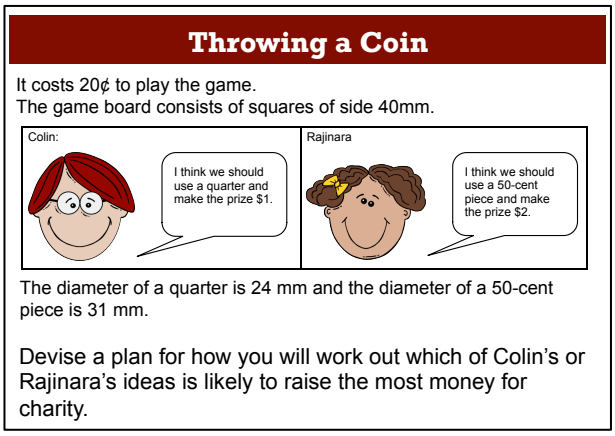
Common issues:	Suggested questions and prompts:
Provides a qualitative comment only For example: The student writes, 'Both coins have a good chance of winning because the squares are bigger than the coins.' (Q1).	• Does one coin have a better chance of winning than the other? How can you tell?
Makes incorrect assumptions For example: The student assumes that the larger the coin the more chance it has of winning (Q1). They may quantify this using the dimensions of the coins, for example. (Quarter has probability 24/40; 50-cent piece has probability 31/40.) Or: Student assumes weight of a coin is a factor.	• What are the rules of the game? • What factors will affect the probability that the coin lands entirely within a square/crosses or touches a line? • Do all coins land on the board? [Yes.] How does weight influence the probability of a coin landing entirely within a square?
Does not answer probabilistically For example: The student writes 'Colin will make $1 - $0.20 = $0.80 whereas Rajinara will make $2 - $0.20 = $1.80' (Q2).	• Will they make that every time? Why / Why not? • What factors will affect the probability that the coin lands entirely within a square/crosses or touches a line? • Can you design an experiment that would help you estimate the probability of each coin winning?
Fails to describe an experiment (Q2) For example: The student writes, 'It depends on how many times someone wins with that coin.'	• Can you design an experiment that would help you estimate the probability of each coin winning?
Confuses the purpose of the game For example: The student focuses on seeing who would 'win' the game with no regard for the charity money. Or: The student does not realise that they were to choose ONE coin to help Colin and Rajinara create the design of the game**.**	• What is your task? • What decisions do Colin and Rajinara need to make? • Remember that Colin and Rajinara want to make money *for charity*.
Designs an experiment with an insufficient number of trials (Q2) For example: The student writes, 'Throw each coin 10 times and count how many times you win'.	• How many trials would you need to get a reasonable estimate for the probability? • What will happen to the probability estimate the more trials you perform?
Description contains insufficient detail For example: The student writes, 'Throw each coin lots of times and see how often you win.' (Q2). Or: Student does not keep track of throw results. Or: Student writes, 'You would need to work out the probability of the coin landing in a square.' (Q3).	• Can you be more specific? How many times would you throw each coin? Why? • What information should you collect? • What would you do with the results? Make up some data to show me. • How could you work out the probability of the coin landing in a square, without carrying out an experiment?

SUGGESTED LESSON OUTLINE

Reviewing individual solutions to the problem (10 minutes)

Give each student a mini-whiteboard, pen, and eraser.

Display Slide P-2 of the projector resource:



Return students' work on the *Throwing a Coin* task.

If you have not added questions to individual pieces of work, either give each student a printed version of your list of questions with the questions relating to their work highlighted or write your list of questions on the board so that students can select questions from the board that are appropriate to their own work.

> *I have looked at your work and have some questions I would like you to think about.*
>
> *On your own, carefully read through the questions I have written.*
>
> *I would like you to use the questions to help you to think about ways of improving your work. Use your mini-whiteboards to make a note of anything you think will help to improve your work.*
>
> *You will be sharing these notes with a partner later on.*

If mini-whiteboards are not available, students may want to use the back of their assessment task or a clean piece of paper to jot down their ideas about ways to improve their work.

This is an opportunity for students to review their own work before working collaboratively to determine which of the two game designs (Colin's or Rajinara's) is likely to result in the most money being raised for charity.

Collaborative small-group work: *Agreeing a Strategy* (15 minutes)

Organize the class into groups of two or three students.

Give each group a blank piece of paper.

> *In your group, you are going to devise a plan for how you will go about working out whose idea will raise the most money for charity.*

Students should use what they have learned from reviewing their individual work on the assessment task to help them to decide on and plan their method.

Slide P-3 summarizes how students should work together:

Agreeing a Strategy

1. Take turns to share with your partner(s) your individual work on the task and the notes you made on how it could be improved.
2. Listen carefully to each other, asking questions if you don't understand.
3. Agree together on a strategy for working out which of Colin's or Rajinara's ideas is likely to raise the most money for charity.
4. Write a plan of what you intend to do clearly on your piece of paper.

Collaborative small-group work: *Implementing the Strategy* (20 minutes)

Give each group of students a copy of the *Game Board* and some plain paper to work on.

> *In your groups you are now going to implement your strategy to determine which of Colin's or Rajinara's ideas is likely to raise the most money for charity.*

Display and explain Slide P-4 on how students should work together on implementing their strategy:

Implementing the Strategy

1. Use your agreed strategy to work out which game design is likely to raise the most money for charity.
2. If you are carrying out an experiment, make sure you:
   a. Estimate the probabilities that each coin will win.
   b. Use the probabilities to estimate which game design will raise the most money.
3. If you have chosen to calculate the probabilities without doing an experiment, explain your work carefully.
4. Decide which of the two ideas Colin and Rajinara should go with.

Explain that if a coin does not land on the board the throw is discounted and the player can throw the coin again, 'for free'.

You have two tasks during the group work: to make a note of student approaches to the task and to support students working as a group.

Make a note of student approaches to the task

Listen and watch students carefully. In particular, notice how students make a start on the task, where they get stuck and how they overcome any difficulties.

Do they throw coins and collect data to help them with their estimate of the probability of winning or do they take a more theoretical approach? If they carry out an experiment, how many times do they throw the coin? What do they do with the results once they have collected some data? Do they consider the effect of the size of the coin compared with the size of the squares on the game board on a person's decision on whether or not to play the game? When considering a theoretical model, how do students calculate the probability that each coin will win a prize?

Support students working as a group

As students work on the task, support them in working together. If you notice that they are not working collaboratively on the task, ask students in the group to explain what someone else in the group has done or said.

What approach are you taking to the problem? Why?

Neil, Sam thinks Rajinara's idea will result in more money being raised for charity, how has he come to that conclusion?

If students have carried out an experiment to estimate the probability of each coin winning, ask them how they could use their results to decide which design would make most money for the charity.

Encourage **all** students to calculate an estimate for the probability, without doing an experiment.

If students have considered both an experimental and theoretical approach, ask them if the game design could be improved further to raise even more money for charity. For example, what would be the effect of changing the cost of playing the game?

Extending the lesson over two days

If you are taking more than one day to complete the unit then you may want to end the first lesson here. At the start of the second day, give students the opportunity to familiarize themselves with their collaborative work on the task, before moving on to the collaborative analysis of sample responses.

Collaborative analysis of *Sample Responses to Discuss* (30 minutes)

Give each group a copy of the two *Sample Responses to Discuss*.

Now you are going to look at some other students' strategies.

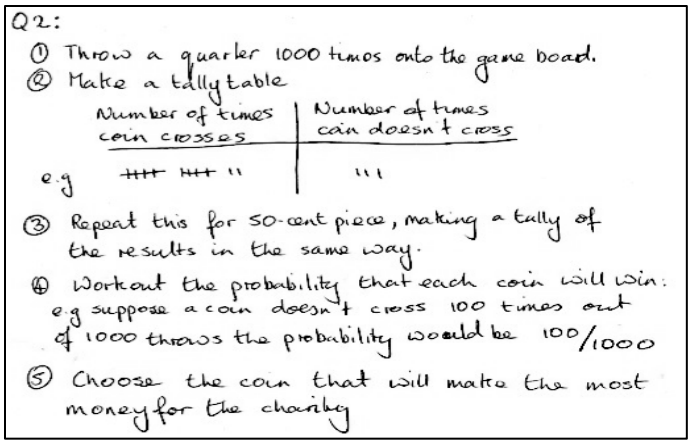
Display and explain to students Slide P-5 of the projector resource:

Sample Responses to Discuss

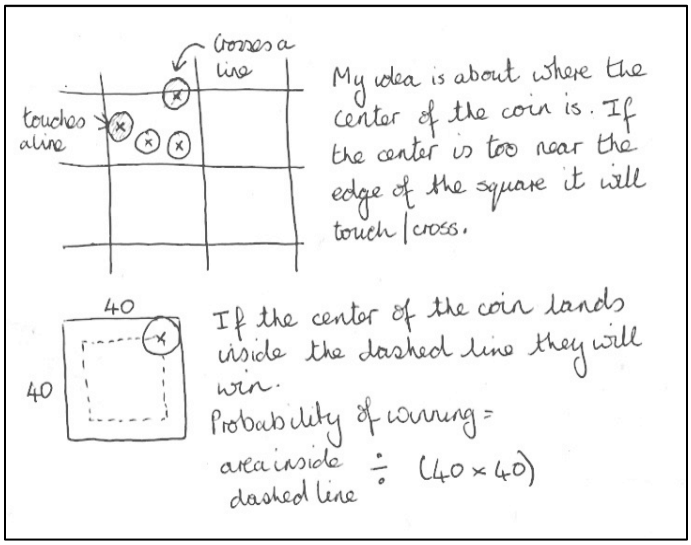
1. Read each plan carefully and try to understand what they intended to do.
2. Think about ways their plan could be improved, taking turns to explain your thinking to your partner(s).
3. Listen carefully to each other, asking questions if you don't understand.
4. When your group has come to a consensus, write your answers to the questions underneath the work.

This task gives students an opportunity to evaluate two possible approaches and to notice any differences and/or similarities with their own work. Students should thoughtfully answer the questions below each piece of sample student work and be encouraged to think carefully about ways in which the work could be improved. Encourage students to focus on evaluating the strategies and math in the student work.

Jasper takes an experimental approach and suggests throwing each of the two coins 1000 times. This would take a very long time. (Students might like to estimate about how long that would take). He does not explain why he chose 1000. He does not say how he would use the probability to determine which coin/prize combination would raise the most money for charity.



Shan takes a theoretical approach, looking at which part of the square the center of the coin must land in order for no part of the coin to touch or cross a line. She does not explain why she focuses on the center of the coin, although this is a good idea. Shan has correctly drawn a square locus for where the center of the coin must be. She has stated that if the center lands inside the dashed line they will win and she will work out this area. Shan correctly states the formula for figuring out the probability of winning. She does not explain how she will use the probability to solve the problem.



Collaborative small-group work: review (20 minutes)

Now that you have looked at the sample student responses, you are going to review your work.

Display and explain Slide P-6 of the projector resource:

Review

1. Review your method for working out which of Colin's and Rajinara's ideas would result in raising the most money.
2. If your strategy was to carry out an experiment, can you use a non-experimental approach to check your work?
3. If you did not carry out an experiment, are you happy with your strategy and solution?
4. Once you are happy with your decision about which idea will raise the most money for charity make sure you can justify your conclusions.

Ask questions that help students relate what they have done to the sample student work:

Have you taken an approach that is similar to that taken in one of the sample pieces of work? Which one? Why?

Did you learn anything from the sample work that might help you now? What is it?

Can you improve on the ideas you had before and the ones seen in the sample work?

How could you improve your work/check your solution?

Can you justify your method and the conclusion you have come to?

Students that have used an experimental approach to estimate the probability of each coin winning might want to use a theoretical approach and check this against their experimental result.

Whole-class discussion: comparing different approaches (15 minutes)

Hold a brief whole-class discussion to consider the different approaches seen and used within the class. Students should be encouraged to consider the advantages and disadvantages of the different possible methods.

Can you explain Jasper's/Shan's approach in your own words?

What did you like/dislike about Jasper's/Shan's approach?

Did anyone use a similar approach to Jasper or Shan? Can you explain what you did?

Did anyone take a totally different approach?

What do you think is the best way to tackle this problem? Why?

What effect did looking at the sample responses have on your work?

What advice would you give to Colin and Rajinara?

Some students may have developed the task further, considering the effect of changing the cost of playing the game and/or the prize money amount.

Follow-up lesson: individual reflection (15 minutes)

Give each student a copy of the *How Did You Work?* questionnaire. Ask students to spend a couple of minutes, individually, answering the questions.

Think carefully about your work this lesson and the different methods you have seen and used.

On your own, answer the review questions as carefully as you can.

Some teachers give this as homework.

SOLUTIONS

Assessment task: *Throwing a Coin*

Q1 A quarter is more likely to win a prize than a 50-cent piece, as it is smaller so the probability that it will land entirely within a square will be greater.

Q2 Students are likely to suggest throwing each coin a certain number of times and seeing what proportion of those times it lands completely inside a square. Provided they used a reasonably large number of trials, this would give a good approximation to the probability of a coin landing completely inside a square. They can use the relative frequency of the coin landing entirely within a square as an estimate for the probability.

Suppose they estimate that a coin will win with probability p.
This would mean that in 1000 throws we would expect the player to win $1000p$ times.
Each throw costs 20¢, so the 1000 throws will cost \$200.

For the quarter, the total winnings for the player will be: \$1000p - \$200
For the 50¢ piece, the total winnings for the player will be: \$2000p - \$200

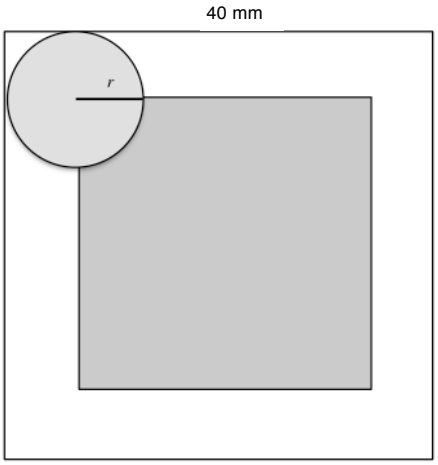
Q3 A coin wins a prize if the center of the coin lies within the shaded square, side 40 mm.

Suppose a coin has a radius of r or diameter d.

The probability that the coin will win a prize (assuming that the coin comes to an abrupt halt at the edge of the board and does not bounce back)

= area of shaded square ÷ area of grid square.

$$= \frac{(40-2r)^2}{40^2} = \frac{(40-d)^2}{40^2}$$



For the quarter the probability of a win

$$= \frac{(40-24)^2}{40^2} = 0.16$$

For the 50-cent coin the probability of a win

$$= \frac{(40-31)^2}{40^2} = 0.05 \text{ (2 d.p.)}$$

If the game is played 1000 times,

the quarter will win (for the players): \$1000 x 0.16 - \$200 = -\$40
the 50-cent piece will win (for the players): \$2000 x 0.05 - \$200 = -\$100

The negative here means that the players will lose, so the charity will gain.

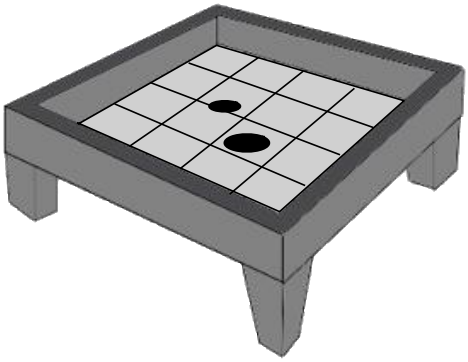
So Rajinara's suggestion is expected to raise more money for charity based on the theoretical probability of winning.

Throwing a Coin

Colin and Rajinara are designing a game to raise money for charity. It costs 20¢ to play.

They have made a game board that consists of a square grid with squares of side 40mm.

They have also agreed on the basic rules of the game.



Rules of the game:

Throw the coin onto the game board.

If it lands entirely within a square, not crossing or touching any lines, you win a prize.

If the coin touches or crosses a line, you lose.

Colin and Rajinara disagree on some of the other design features of the game:

Colin:

I think we should use a quarter and make the prize $1.

Rajinara

I think we should use a 50-cent piece and make the prize $2.

The diameter of a quarter is 24 mm and the diameter of a 50-cent piece is 31 mm.

1. Which coin is most likely to win a prize? Why is this?
2. Colin and Rajinara want to work out which game design will raise the most money for charity. Describe carefully how they could do a simple experiment to find out.

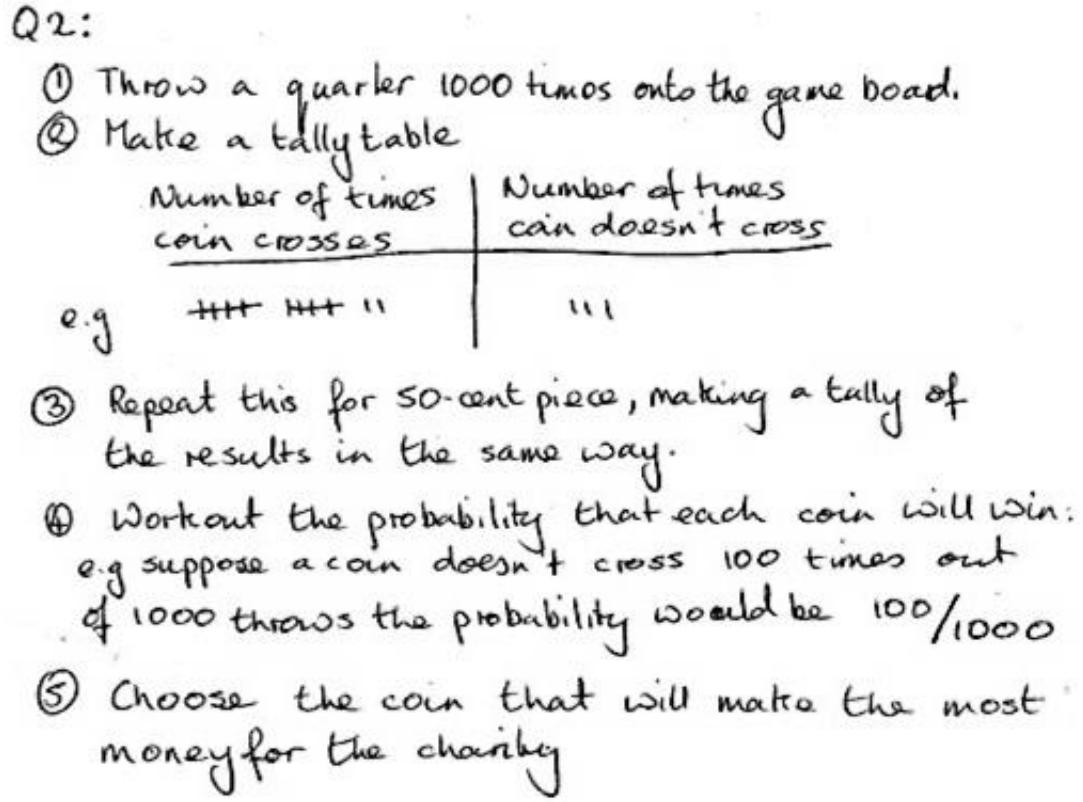
   In your description, explain how they can:

   - estimate the probabilities that each coin will win;
   - use these probabilities to estimate which game design will raise the most money.

   Make up some data if this helps you to explain what to do.
3. How could you calculate the probabilities that each coin will win a prize, without doing an experiment? Write down your ideas.

Game Board

Sample Responses to Discuss: Jasper



Q2:

① Throw a quarter 1000 times onto the game board.

② Make a tally table

	Number of times coin crosses	Number of times coin doesn't cross
e.g	𝍸 𝍸 ||	|||

③ Repeat this for 50-cent piece, making a tally of the results in the same way.

④ Workout the probability that each coin will win: e.g suppose a coin doesn't cross 100 times out of 1000 throws the probability would be 100/1000

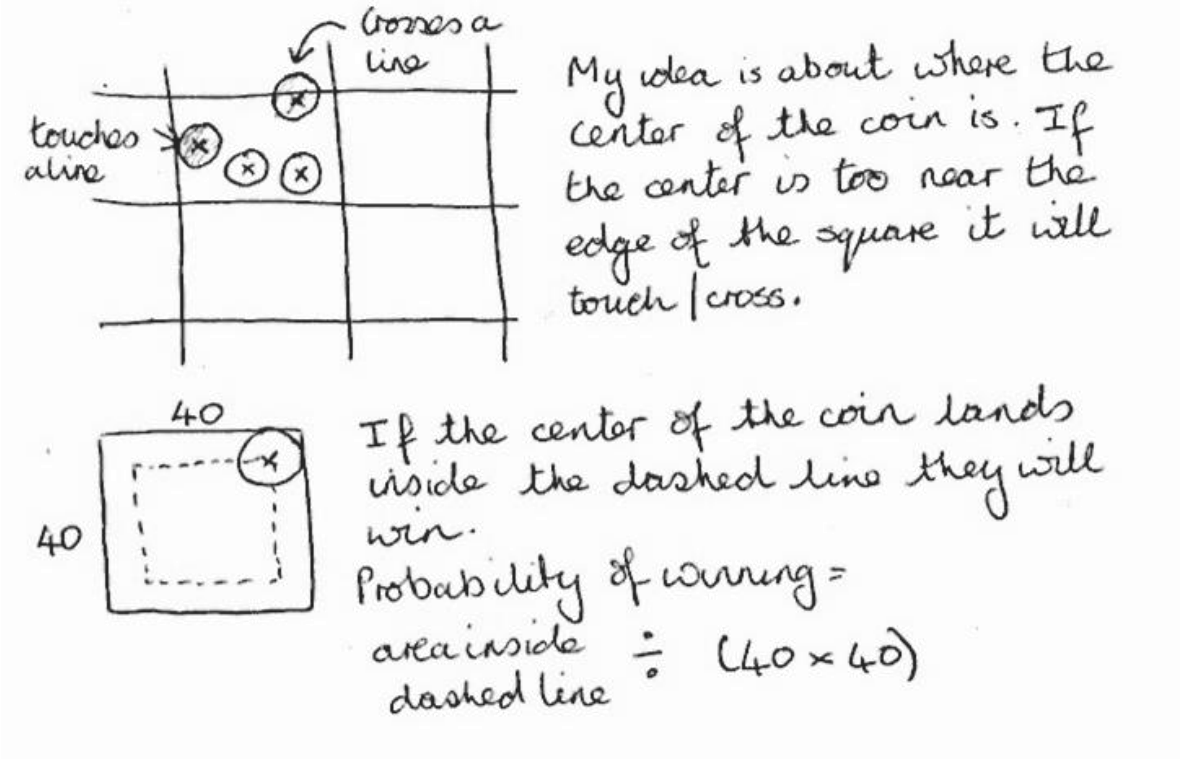
⑤ Choose the coin that will make the most money for the charity

1. Is Jasper's plan complete and correct? Why / Why not?

2. What mathematics is Jasper using? Does it make sense to use this type of mathematics? Why / Why not?

3. What other mathematics could/should Jasper use?

Sample Responses to Discuss: Shan



1. Is Shan's plan complete and correct? Why / Why not?

2. What mathematics is Shan using? Does it make sense to use this type of mathematics? Why / Why not?

3. What other mathematics could/should Shan use?

How Did You Work?

Check the boxes, circle an option and complete the sentences that apply to your work.

1. Our group strategy was better than my own. **Yes/ No**

   This is because .....

   .....

2. Our strategy was similar to one of the sample responses. ☐ **OR** Our strategy was different from **both** sample responses. ☐

   Our strategy was similar to ..... *(add name of sample response)*

   I prefer **our strategy / the sample response strategy** (*circle)*

   because .....

   .....

   .....

   .....

   Our strategy was different from both sample responses

   because .....

   .....

   .....

   .....

3. I made some assumptions. ☐

   My assumptions were: .....

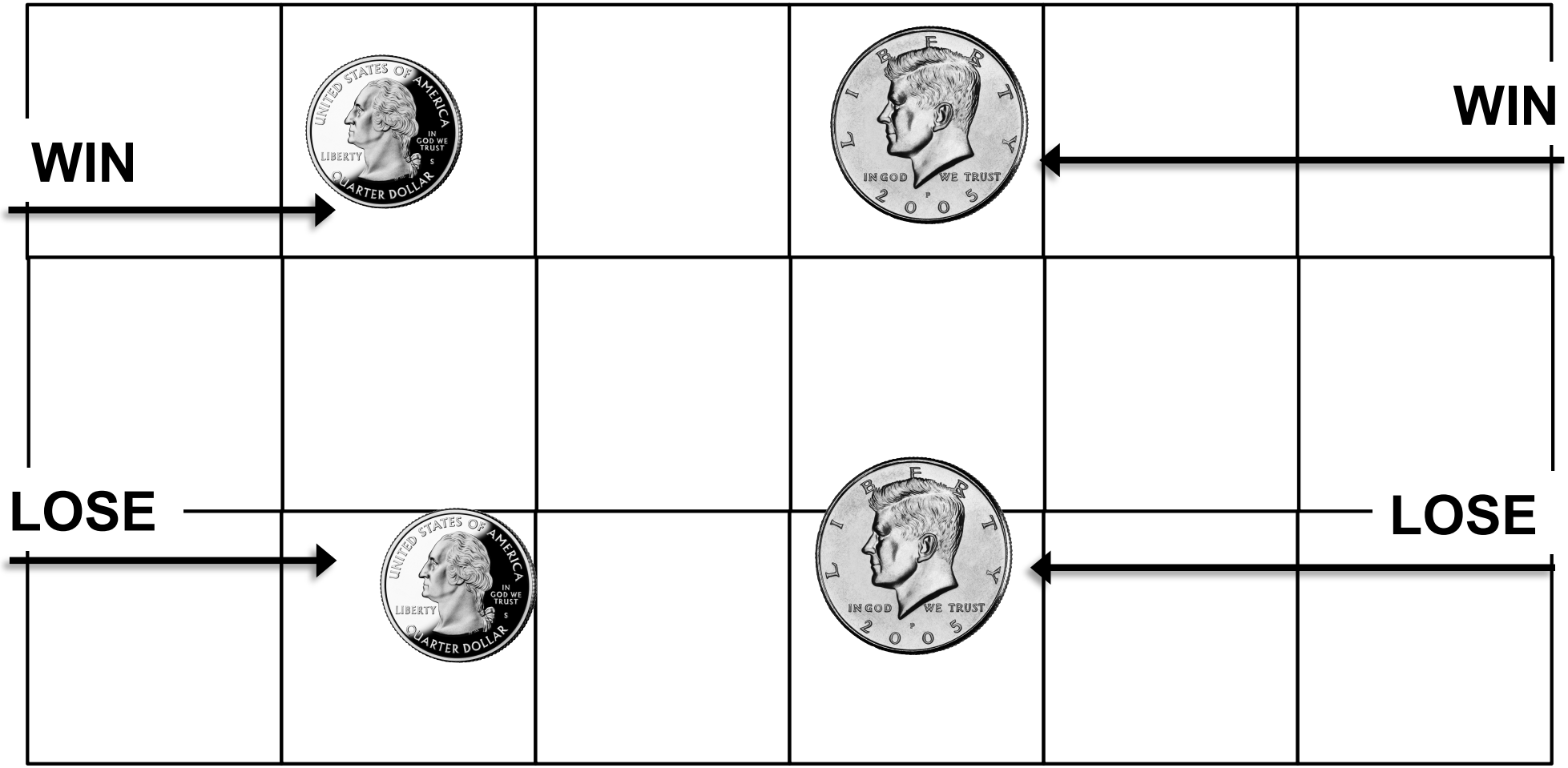
   .....

4. The difficulties I think a student new to this task would have are:

   .....

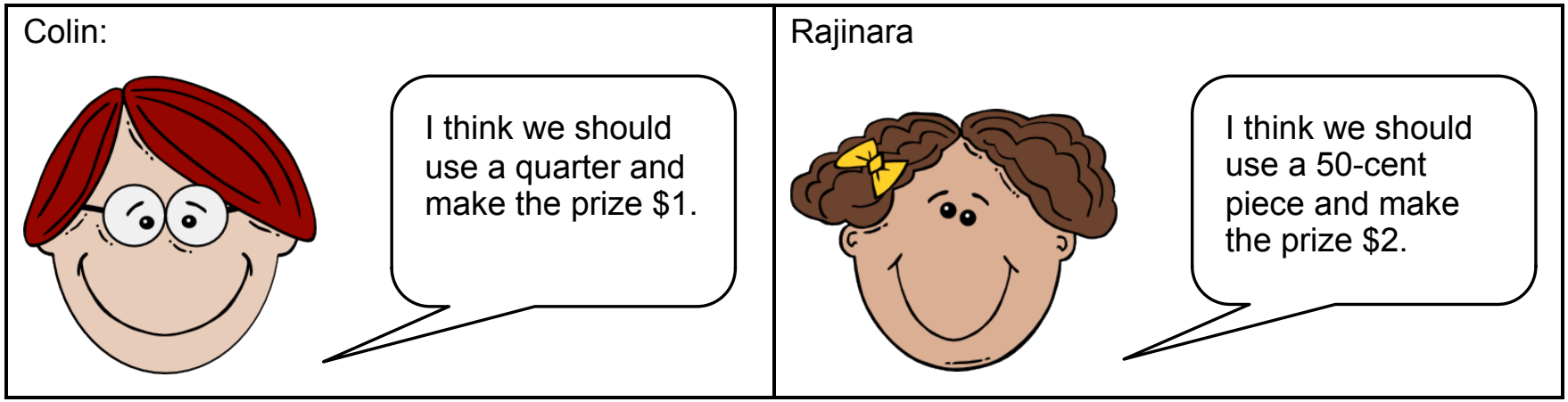
   .....

Rules of the Game



Throwing a Coin

It costs 20¢ to play the game.
The game board consists of squares of side 40mm.



The diameter of a quarter is 24 mm and the diameter of a 50-cent piece is 31 mm.

Devise a plan for how you will work out which of Colin's or Rajinara's ideas is likely to raise the most money for charity.

Agreeing a Strategy

1. Take turns to share with your partner(s) your individual work on the task and the notes you made on how it could be improved.

2. Listen carefully to each other, asking questions if you don't understand.

3. Agree together on a strategy for working out which of Colin's or Rajinara's ideas is likely to raise the most money for charity.

4. Write a plan of what you intend to do clearly on your piece of paper.

Implementing the Strategy

1. Use your agreed strategy to work out which game design is likely to raise the most money for charity.

2. If you are carrying out an experiment, make sure you:
   a. Estimate the probabilities that each coin will win.
   b. Use the probabilities to estimate which game design will raise the most money.

3. If you have chosen to calculate the probabilities without doing an experiment, explain your work carefully.

4. Decide which of the two ideas Colin and Rajinara should go with.

Sample Responses to Discuss

1. Read each plan carefully and try to understand what they intended to do.

2. Think about ways their plan could be improved, taking turns to explain your thinking to your partner(s).

3. Listen carefully to each other, asking questions if you don't understand.

4. When your group has come to a consensus, write your answers to the questions underneath the work.

Review

1. Review your method for working out which of Colin's and Rajinara's ideas would result in raising the most money.

2. If your strategy was to carry out an experiment, can you use a non-experimental approach to check your work?

3. If you did not carry out an experiment, are you happy with your strategy and solution?

4. Once you are happy with your decision about which idea will raise the most money for charity make sure you can justify your conclusions.

Sample Responses to Discuss: Jasper

Q2:

① Throw a quarter 1000 times onto the game board.

② Make a tally table

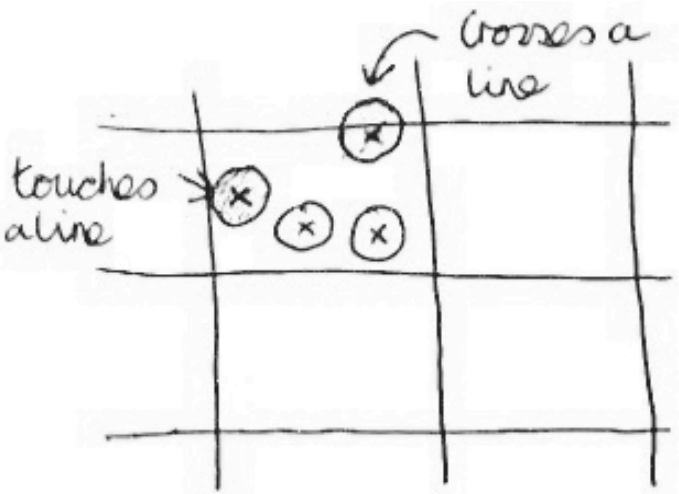
	Number of times coin crosses	Number of times coin doesn't cross
e.g	𝍸 𝍸 ||	|||

③ Repeat this for 50-cent piece, making a tally of the results in the same way.

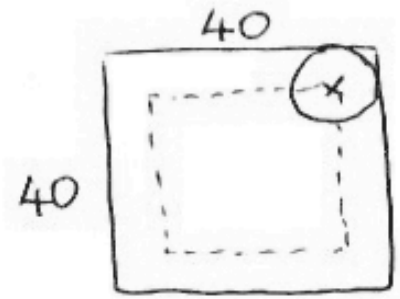
④ Work out the probability that each coin will win:
e.g suppose a coin doesn't cross 100 times out of 1000 throws the probability would be 100/1000

⑤ Choose the coin that will make the most money for the charity

Sample Responses to Discuss: Shan



My idea is about where the center of the coin is. If the center is too near the edge of the square it will touch / cross.



If the center of the coin lands inside the dashed line they will win.

Probability of winning =

area inside dashed line ÷ (40 × 40)

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org