PROBLEM SOLVING

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Designing a 3D Product in 2D: *A Sports Bag*

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Designing a 3D Product in 2D: *A Sports Bag*

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Recognize and use common 2D representations of 3D objects.
- Identify and use the appropriate formula for finding the circumference of a circle.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 1, 2, 3, and 6:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
8. Look for and express regularity in repeated reasoning.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

7.G: Draw, construct, and describe geometrical figures and describe the relationships between them.
Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

7.EE: Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

INTRODUCTION

The lesson unit is structured in the following way:

- Before the lesson, students attempt the *Designing a Sports Bag* task individually. You review their work and formulate questions that will help students to improve their solutions.
- At the start of the lesson, students respond individually to the questions set. Then in groups, they combine their thinking and work together to produce a joint solution in the form of a poster.
- In the same small groups, students evaluate and comment on sample responses. They identify the strengths and mistakes in these responses and compare them with their own work.
- In a whole-class discussion, students explain and compare the strategies they have seen and used.
- Finally, students reflect on their work and their learning.

MATERIALS REQUIRED

- Each student will need a copy of *Designing a Sports Bag*, some plain paper, a mini-whiteboard, a pen, and an eraser.
- Each small group of students will need a new copy of *Designing a Sports Bag*, some poster paper, a marker, and copies of the *Sample Responses to Discuss*. Provide calculators, rules, and squared paper for students who request them. An optional *Formula Sheet* is available for use as required.

TIME NEEDED

25 minutes before the lesson, a 95-minute lesson (or two 50-minute lessons), and 15 minutes in a follow-up lesson. Timings are approximate. Exact timings will depend on the needs of your class.

BEFORE THE LESSON

Introducing the task: *Designing a Sports Bag* (25 minutes)

Ask the students to do this task, in class or for homework, a day or more before the lesson. This will give you an opportunity to assess their work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.

Before students are given the task, take time to help them to understand the problem context as this may be unfamiliar to some. There are projector resources to help you to do this.

Have you ever seen an item of clothing or a bag being made out of fabric?

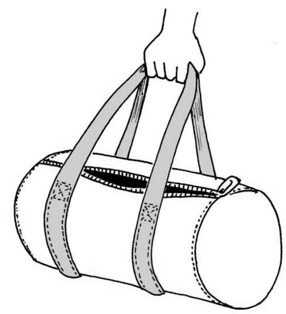
Display Slide 1 of the projector resource.

Suppose you were making this tote bag.

Ignoring the handles, what pieces of fabric would you need to make the body of the bag? [Five rectangular pieces – a front and back (same size), two end panels (same size) and a piece for the base.]

Designing a Sports Bag

You have been asked to design a sports bag:



- The length of the bag will be 20 inches.
- The bag will have circular ends of diameter 11 inches.
- The main body of the bag will be made from three pieces of fabric: a piece for the curved body, and the two circular end pieces.
- When cutting out pieces of fabric for the bag, each piece will need an extra ½ inch all the way around it. This is the seam allowance and allows for the pieces to be stitched together.

1. Make a sketch of the pieces you will need to cut out for the body of the bag. On your sketch, show all the measurements you will need.
2. Suppose you are going to make one of these bags from a roll of fabric 1 yard wide. What is the shortest length of fabric you could cut from the roll? Describe, using words and sketches, how you arrive at your answer.

Introduce the idea of a seam allowance.

Display Slide P-2 of the projector resource.

Pattern pieces are drawn out on paper for each section of the bag, with an extra ½ inch added to each edge to allow for the fabric pieces to be stitched together/hemmed. This is called the seam allowance. (Folding the seam allowance over and stitching along it produces a hem. Once the pieces of fabric have been stitched, the seam allowance is hidden inside the bag.)

Focus students' attention on how different arrangements of pattern pieces use more or less fabric length.

Display Slide P-3 of the projector resource.

The pattern pieces are pinned onto a roll of fabric called a bolt. A bolt will come in a certain width, depending on the type of fabric and so you buy the length of fabric you need off the roll.

Display Slide P-4 of the projector resource.

The aim is to use the smallest length of fabric possible, so before pinning the pattern pieces onto the fabric it is important to think about the most efficient way to arrange them.

Which of these three arrangements will use the smallest length of fabric? [C.]

Spend some time discussing the three arrangements and identify whether some of them could be improved upon, without rearranging every piece. For example, both arrangements A and B can be improved so that less fabric length is required.

Display Slide P-5 of the projector resource.

Once the pattern pieces have been arranged in the most efficient way, they can be pinned onto the fabric and then cut around. The fabric pieces can then be stitched together to make the bag.

Now explain what you are asking students to do. Give each student a copy of the *Designing a Sports Bag* task and some plain paper for them to work on. Display Slide P-6 of the projector resource.

> *In this task you will work on a different bag: a sports bag. You will need to think about the pattern pieces and include seam allowances so that the pieces can be cut from the fabric and stitched together.*
>
> *I would like you to spend 15 minutes on your own, answering the questions as carefully as you can. Show all your work so that I can understand your reasoning.*
>
> *There will be a lesson [tomorrow] that will help you to improve your work.*

It is important that, as far as possible, students are allowed to answer the questions without assistance. If students are struggling to get started, then ask questions that help them understand what is required, and the context of the problem, but make sure you do not do the task for them.

Students who sit together often produce similar solutions so that, when they compare their work, they have little to discuss. For this reason we suggest that when students do the task individually you ask them to move to different seats. Then, at the beginning of the formative assessment lesson, allow them to return to their usual seats. Experience has shown that this produces more profitable discussions.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their problem solving strategies.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students at the beginning of the lesson.

Common issues:	**Suggested questions and prompts:**
Draws pattern pieces incorrectly For example: The student has not recognized that the body of the bag is formed from a rectangle.	• Imagine you remove the ends of the bag, and unfold the center. If you lay that piece flat, what shape will it be?
Ignores assumptions and constraints in making the bag For example: The student omits a seam allowance on some edges, such as the length or width of the rectangle. Or: The student adds a seam allowance or seam allowances before calculating the length of the rectangle.	• Have you used all the information in the question? • Where will the sides of the rectangle be attached? How? • Which of these circles is sewn to the length of the rectangle: the inner circle, or the outer? How does your answer affect the length of the rectangle you need to cut?
Does not show any calculations	• How did you calculate the length of the rectangular piece? • Can you explain how you have calculated the outside measurements of the rectangle, including the seam allowance?
Does not work to appropriate degree of accuracy For example: The student works to 4 decimal places, in a context that involves cutting fabric by hand. Or: The student rounds π to 3, reducing the length of the rectangle significantly.	• How accurate do your answers need to be for the pieces to fit? • Do you think it is better to round your answer up or round it down when deciding the lengths to cut? Why?
Does not justify his/her answers adequately For example: The student does not provide a sketch showing seam allowances. Or: The student does not show how the shortest length of fabric needed has been obtained. Or: The student's sketches do not include measurements.	• Explain in writing how you would cut these pieces from a roll of fabric 1 yard wide, wasting the least amount of fabric. • Show me clearly how the different shapes fit on the roll of fabric. Convince me that they fit, including the seam allowances!
Completes the task	• Explain how you can be sure that you have found the best possible arrangement, using the least fabric length. Would your solution change if the fabric were 60 inches wide rather than a yard wide? How?

SUGGESTED LESSON OUTLINE

Reviewing individual solutions to the task (10 minutes)

Give each student a mini-whiteboard, pen, and eraser. Return their work on *Designing a Sports Bag*.

If you have not added questions to students' work, write a short list of your most common questions on the board. Students can then select a few questions appropriate to their own work.

Recall the Designing a Sports Bag problem? What was the task about?

I have had a look at your work and I have some questions.

I would like you to think, on your own, about my questions and how your work could be improved.

Students may want to jot their ideas down as they consider how to improve their work. They can either write directly onto their original work using a different colored pen or record their ideas using their mini-whiteboards.

Collaborative activity: making posters (30 minutes)

Organize students into groups of two or three and give each group a piece of poster paper and a marker pen. Ask students to have another go at the task but, this time, ask them to combine their ideas and use what they have learned to produce a joint solution in the form of a poster.

You each have your own solution and have been thinking about how you might improve this. I want you to share your work with your partner(s). Take turns to explain how you approached the task and how you think you could now improve your solution.

Together in your group, produce a poster that shows a joint solution to the task, which is better than your individual responses. Give clear reasons for your choice of method.

While students work in small groups you have two tasks: to note different approaches to the task and to support student problem solving.

Note different student approaches to the task

Listen and watch students carefully and note different approaches to the task. Do students choose appropriate dimensions when calculating missing measurements? Do they consider all constraints in making the bag? Have they described their method and solution effectively? Are they working to an appropriate degree of accuracy? This information will help you to focus a whole-class discussion later in the lesson.

Support student problem solving

Try not to make suggestions that move students towards a particular solution. Instead, ask questions that help students to clarify their thinking. The suggestions in the *Common issues* table may be helpful. If one group is struggling with an issue, you could refer them to a group who are more successful on that issue for help. If the whole-class is struggling on the same issue, you could write a relevant question on the board, or hold a brief whole-class discussion. You may ask them to consider these questions:

In your groups, what have you done individually that you agree on?

Have you used all the information given in the task?

Do your calculations make sense?

The purpose of these questions is to help students to track and review their problem solving strategies. They should be encouraged to give reasons for the choices they make.

Sharing different approaches (10 minutes)

When students have had sufficient time to work on their posters, you may want to hold a whole-class discussion on the method used to produce a group solution. What have students learned by sharing their ideas? How did they decide on their joint solution? Did students check their work? You may want to select a few groups to report back to the rest of the class.

Extending the lesson over two days

If you are taking two days to complete the unit then you may want to end the first lesson here. At the start of the second day, briefly remind students of their previous work before moving on to the collaborative analysis of sample responses.

Collaborative analysis of *Sample Responses to Discuss* (30 minutes)

Give each group of students a copy of each of the *Sample Responses to Discuss*. It may not be appropriate, or there may not be enough time, for all students to analyze all three sample responses. Each response highlights different misconceptions and so, depending on the progress already made on the task, it may be appropriate to issue different sample responses to different groups. For example, groups that have omitted some of the seam allowances could be given Ben's work, while groups that have rounded to an inappropriate degree of accuracy or omitted to check for alternative arrangements could be given Aisha's work.

The whole-class discussion held after the collaborative work should help to inform your decision on whether or not to be selective about which sample responses students are given.

In your groups you are now going to look at some student work on the task. Notice in what ways this work is similar to yours and in which ways it is different.

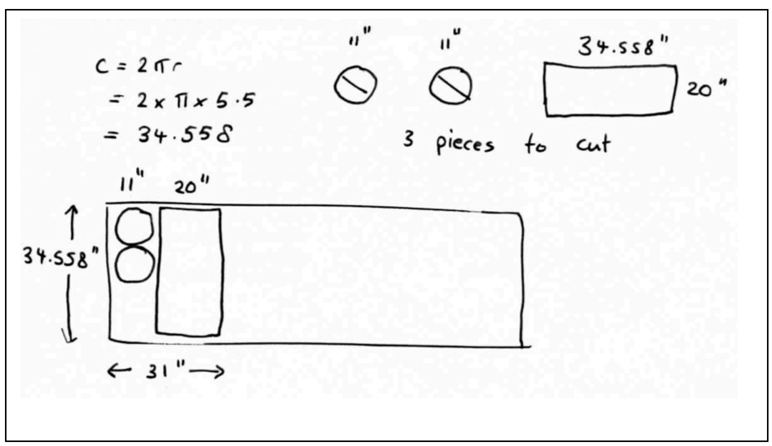
There are some questions for you to answer as you look at the work. You may want to correct the work or add annotations to make it easier to follow.

This task gives students an opportunity to evaluate a variety of possible approaches to the task, without providing a complete solution strategy. Students should, thoughtfully, answer the questions below each piece of sample student work and be encouraged to think carefully about ways in which the work could be improved.

Slide P-7 of the projector resource provides an overview of the questions the students will be answering as they analyze the sample student responses.

Ben draws minimal diagrams for Q1. He shows calculations without explaining their relevance to the diagrams. Ben forgets to add seam allowances to the rectangle and the circle so the measurements he uses are too short.

Ben records calculations correct to three decimal places. Even cutting fabric accurate to the nearest tenth of an inch would be difficult! Rounding to the nearest ½ or ¼ inch is appropriate in this context.



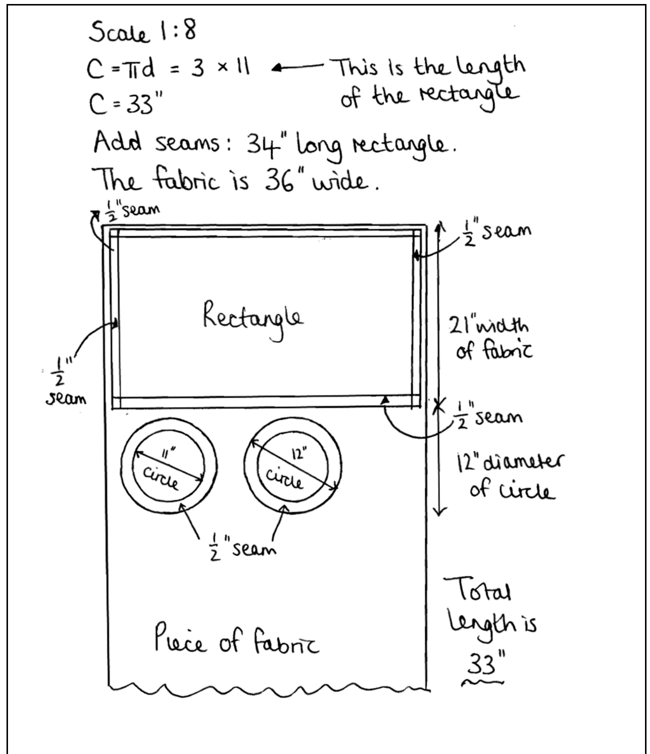
Ben's solution could be improved if he explained his work more clearly, for example, showing the connections between diagrams and calculations. He needs to take seam allowances into account, and work to an appropriate degree of accuracy. He could also provide alternative responses to Q2, to justify his solution.

Aisha has made accurate scale drawings of the pattern pieces, labeled lengths, and shown the scale for Q2, but has not made separate drawings for Q1. She has added appropriate seam allowances.

Aisha explains her calculations and why she is making them. She has labeled the diameter as 11 inches, showing that she knows what the term means.

Aisha does not work to an appropriate degree of accuracy for the context. She rounds π to 3, significantly reducing the length of the rectangle that would be cut. The piece would be too short to sew around the circumference of an 11-inch circle, even including a ½ inch seam allowance at each end.

Aisha provides only one solution to Q2 and so does not show that her solution is optimal. She does not compare alternatives. Aisha could improve her solution by working to an appropriate degree of accuracy and by considering different options in Q2.

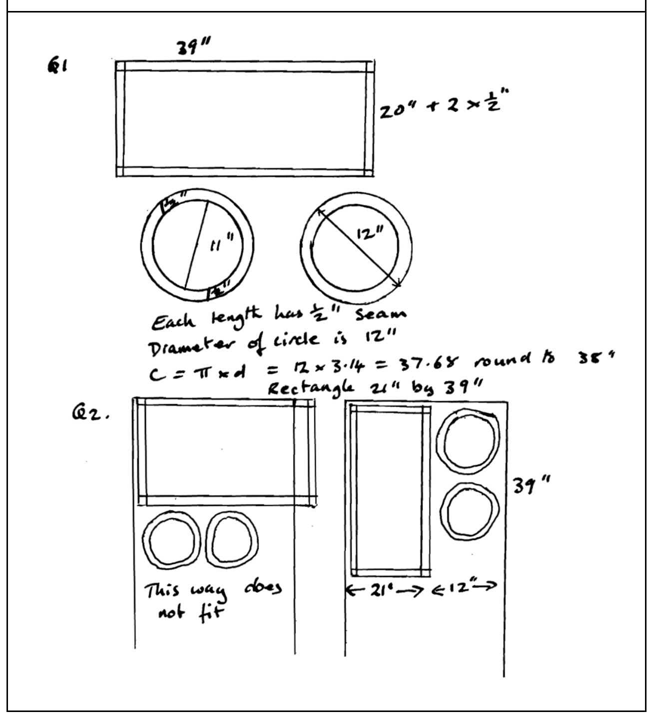


Carlotta correctly draws the pattern pieces, remembering to take account of seam allowances. She works to an appropriate degree of accuracy.

She incorrectly adds seam allowances to the diameter of the circle before calculating the length of the rectangle, so her rectangle is too long and its length will not fit within the width of the fabric.

Carlotta shows she understands the issue of different arrangements of pattern pieces by giving examples. This is an attempt to justify her solution to Q2 by showing one solution and a non-solution. Her answer is incorrect because she has made the rectangle of fabric too long.

Her solution could be improved by better explanation in Q1 and by excluding the seam allowances when calculating the length of the rectangle.



Whole-class discussion: comparing different solution methods (15 minutes)

Discuss the different approaches used in the sample work and ask students to comment on what they have noticed. You may also want to compare students' own work with the sample student responses.

Which student's work was the easiest to follow? Why?

Which piece of sample work does the best job of justifying their response to Q2? How could it be improved further?

In all three of the sample responses the students use rounding at some point in their solution. Were any of the rounding methods appropriate for the problem?

Did any group use a similar method to Ben, Aisha or Carlotta? What was the same about the work? What was different?

In what ways did analyzing the responses help to identify errors in your own work?

You may want to use Slides P-8, P-9, and P-10 of the projector resource and the questions in the *Common issues* table to support this discussion.

Follow-up lesson: individual reflection (15 minutes)

Give each student a copy of the questionnaire *How Did You Work?*

Think carefully about your work on this task and the different methods you have seen and used.

On your own, answer the review questions as carefully as you can.

Some teachers give this task as homework.

SOLUTIONS

1. The student's sketch should show two circular ends and a rectangular piece. Each piece should show seam allowances of ½ inch all the way around.

   The circular ends are of diameter 11 inches with ½ inch added all around, to give 12 inches as the total diameter of the piece cut. However, the length of the rectangular piece is the same as the circumference of the circular end *without* seam allowances:

   $C = \pi d = \pi \times 11 = 34.5564... \approx 34.75''$

   Alternatively, if *d = 11 inches, r = 5.5 inches.* Then $C = 2\pi r = 2 \times \pi \times 5.5 = 34.5564... \approx 34.75''$.

   Rounding up is required, because the rectangle must be *at least* as long as the circumference in this context. Rounding to the nearest ¼ inch or ½ inch is appropriate for the context of cutting fabric by hand.

   Some students may add ½ inch or 1inch seam allowance before calculating the circumference, giving an incorrect measurement for the length of the rectangle. The length of the rectangular piece is the same as the circumference of the circular end *without* seam allowances; that length is sewn to the circumference.

   The rectangular piece thus measures:

   $(20 + 2 \times \frac{1}{2})$ by $(34.75 + 2 \times \frac{1}{2})$ or 21 inches by 35.75 inches.

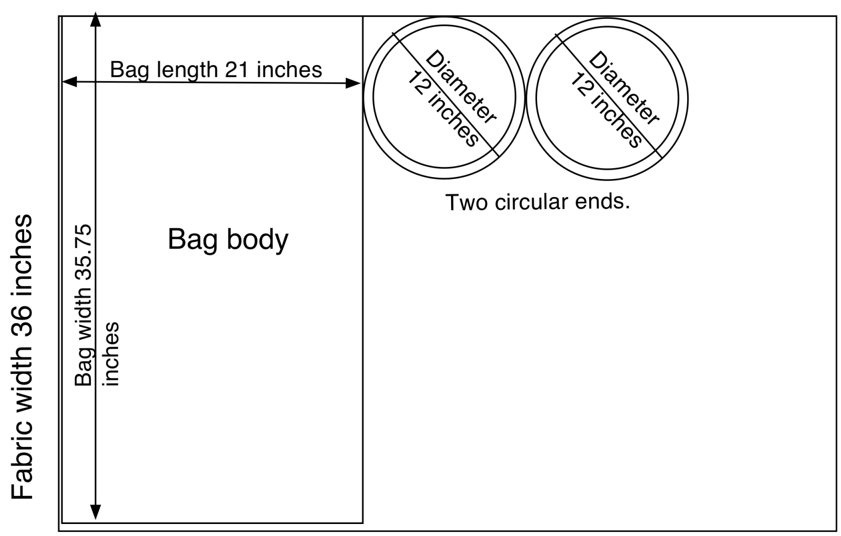
   All diagrams should be labeled with the measure of the length, what the length is, and show seam allowances.

   Calculations should be recorded and the reasons for the calculations explained.

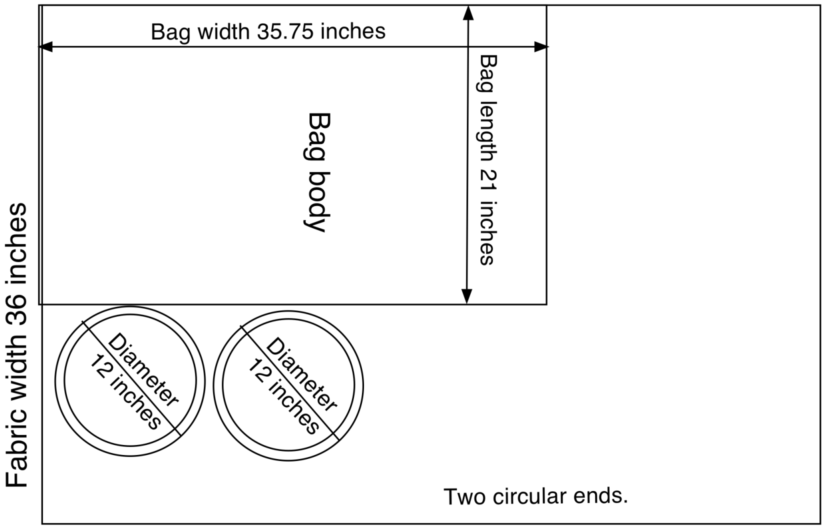
2. The student should draw at least one sketch, showing how the pieces can be arranged. S/he should give a suitable length for the fabric that is required. Ideally, this would include small gaps to ease cutting out.

   A full solution will compare some different arrangements and explain which are optimal. There should be evidence that alternatives have been considered and dismissed. Diagrams should be fully labeled and calculations explained, as above.

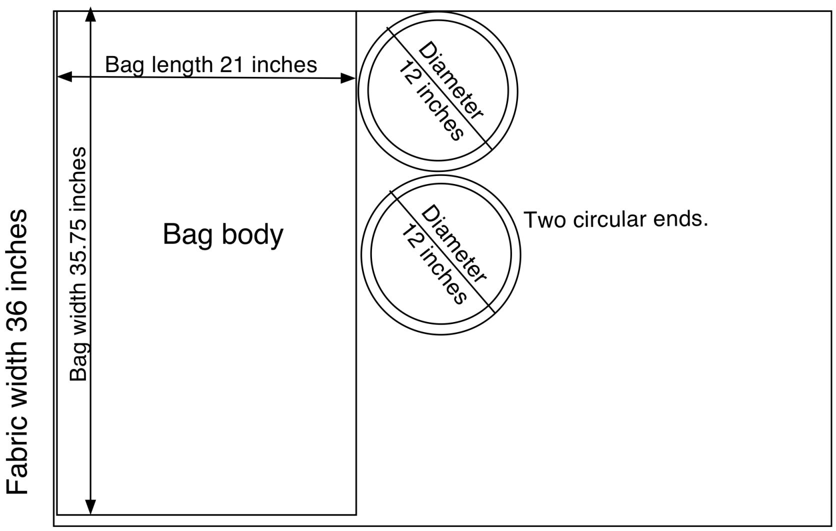
   In this orientation, 45 inches of fabric are needed.



By rotating the bag body through 90 degrees, the circular end pieces fit underneath. For this orientation, 36 inches of fabric are needed (rounded to the nearest purchasable inch).



Rotating the fabric and putting the circles vertically in line with each other requires only 33 inches of fabric. This seems to be optimal. You might push students to say whether, and how, they can be sure of this result.

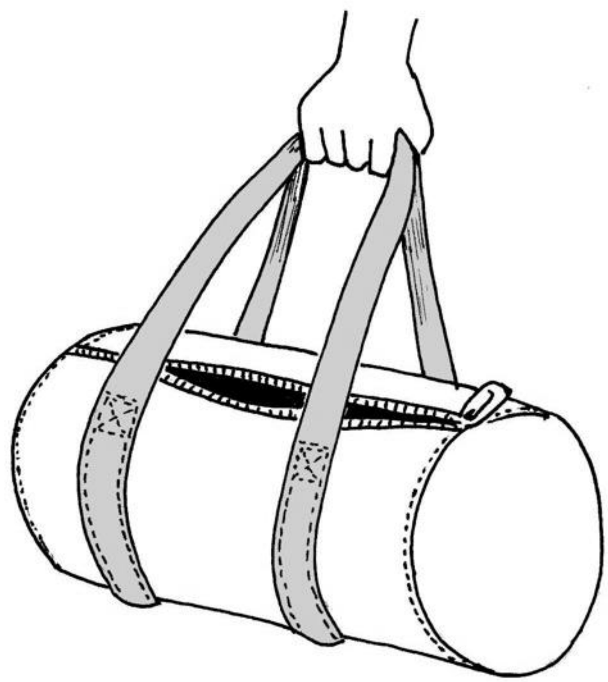


Working systematically through different ways of placing the pattern pieces, identifying congruent arrangements, produces an adequate, exhaustive proof of the result.

Some students might like to figure out whether cutting the fabric for the body of the bag on the cross (at an angle of 45 degrees to the top edge of the bolt) would change the optimal solution.

Designing a Sports Bag

You have been asked to design a sports bag:



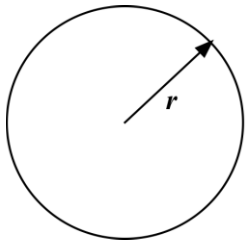
- The length of the bag will be 20 inches.
- The bag will have circular ends of diameter 11 inches.
- The main body of the bag will be made from three pieces of fabric: a piece for the curved body, and the two circular end pieces.
- When cutting out pieces of fabric for the bag, each piece will need an extra ½ inch all the way around it. This is the seam allowance and allows for the pieces to be stitched together.

1. Make a sketch of the pieces you will need to cut out for the body of the bag. On your sketch, show all the measurements you will need.

2. Suppose you are going to make one of these bags from a roll of fabric 1 yard wide. What is the shortest length of fabric you could cut from the roll? Describe, using words and sketches, how you arrive at your answer.

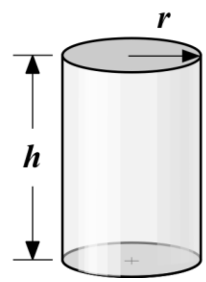
Formula Sheet

You may find some of these formulas helpful:



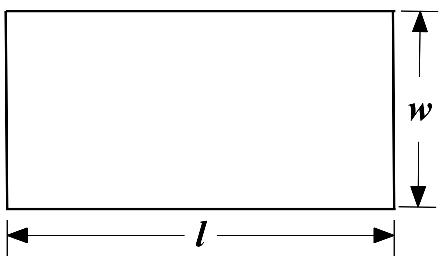
Area of a circle: πr^2

Circumference of a circle: $2\pi r$



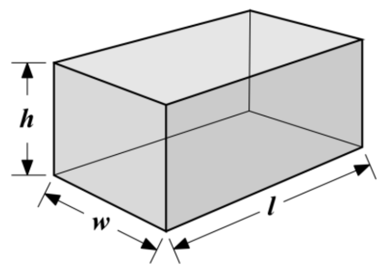
Volume of a cylinder: $\pi r^2 h$

Curved surface area of a cylinder: $2\pi rh$



Area of a rectangle: lw

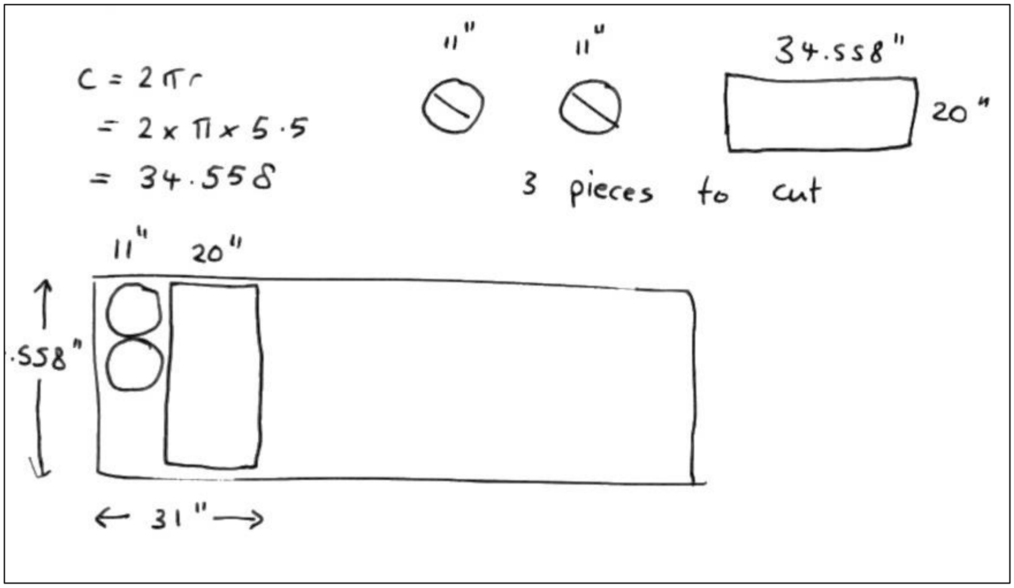
Perimeter of a rectangle: $2(l+w)$



Volume of a right rectangular prism: lwh

Surface area of a right rectangular prism: $2(lw+lh+wh)$

Sample Responses to Discuss: Ben



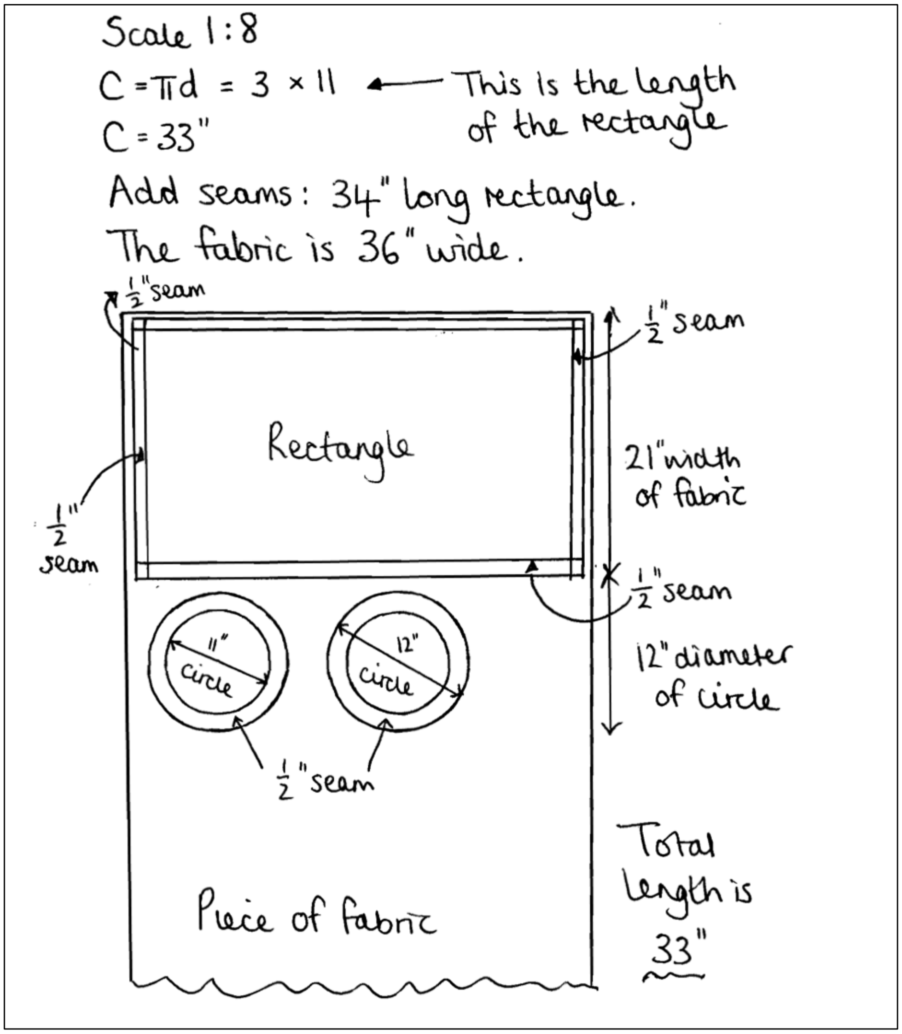
What is missing from Ben's diagrams?

What mistakes has Ben made?

In what ways could Ben's work be improved?

To help you to understand Ben's work, what question(s) could you ask him?

Sample Responses to Discuss: Aisha

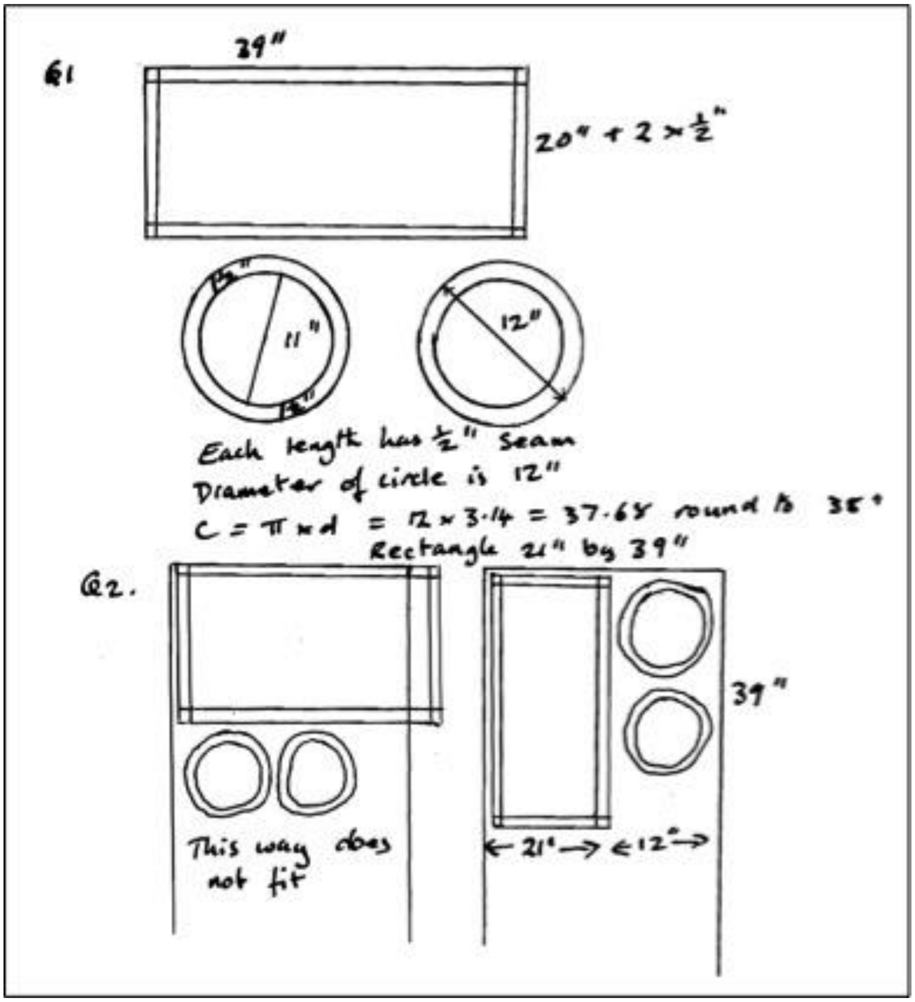


What method has Aisha used? Explain how you know.

What mistakes has Aisha made?

In what ways could Aisha's work be improved?

Sample Responses to Discuss: Carlotta



Has Carlotta worked to an appropriate degree of accuracy? Explain your answer.

What mistakes has Carlotta made?

In what ways could Carlotta's work be improved?

How Did You Work?

Tick the boxes and complete the sentences that apply to your work.

1. The length of fabric needed in my individual work was ☐ inches.
2. The length of fabric needed in our group work was ☐ inches.

   My individual work/our group work was better because ....

   ....

   ....

   ....

3. We checked we had found the shortest length of fabric ☐

   We checked by .... We could have checked by ....

   ....

   ....

   ....

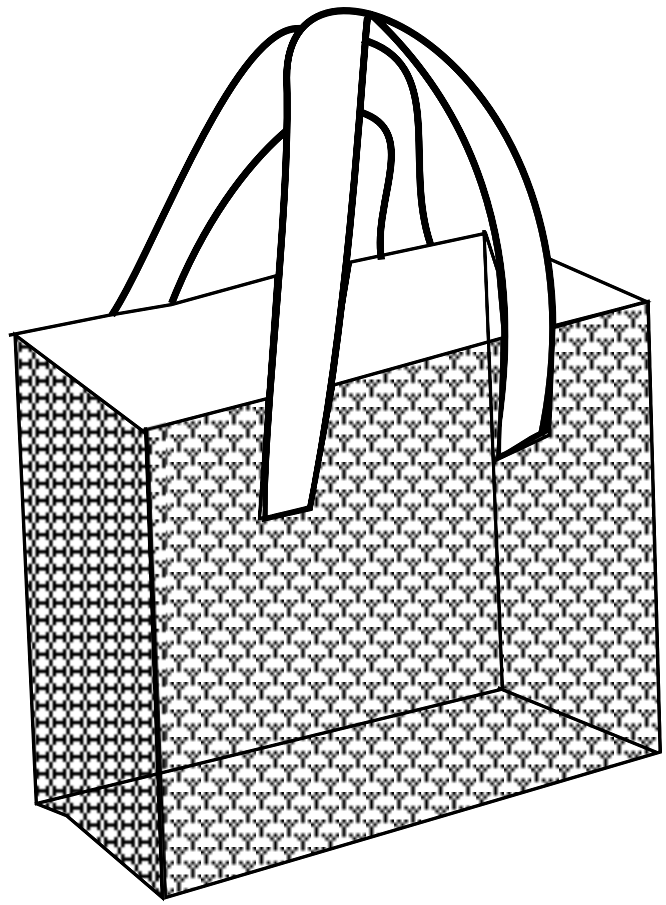
4. In our method we assumed that: ....

   ....

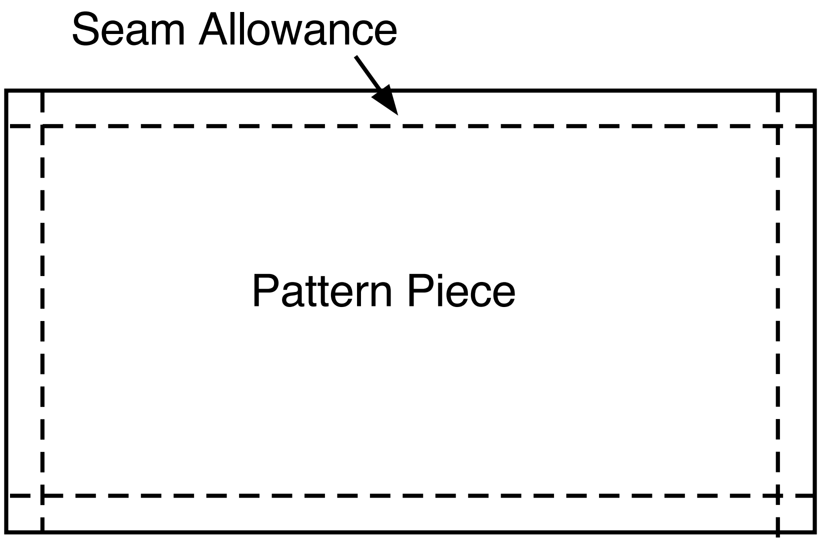
   ....

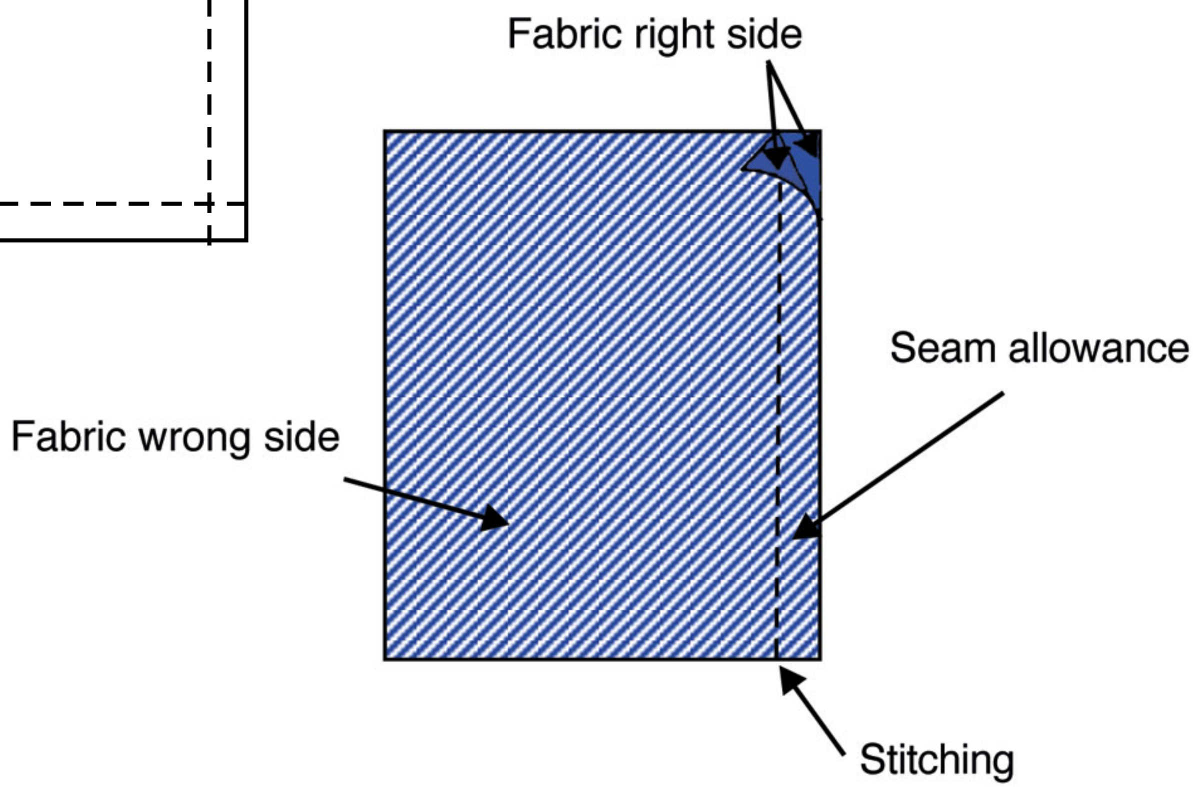
   ....

An Example: Making a Tote Bag



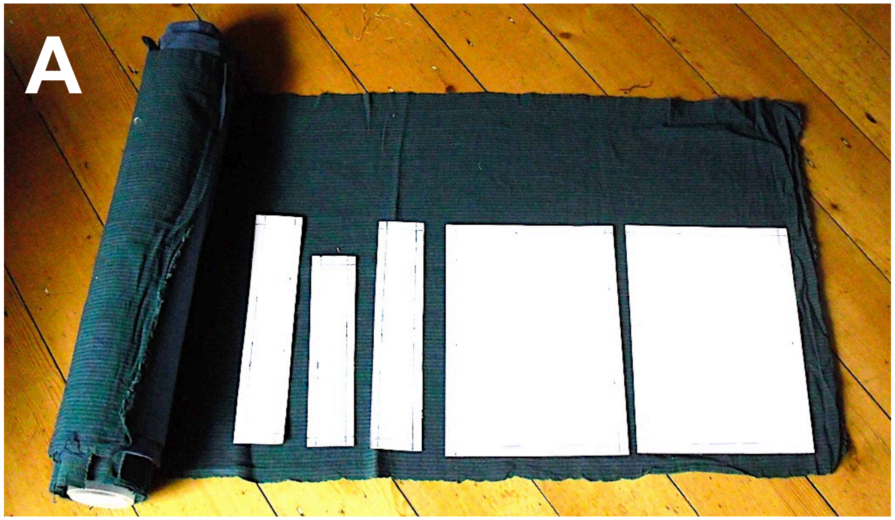
Seam Allowance



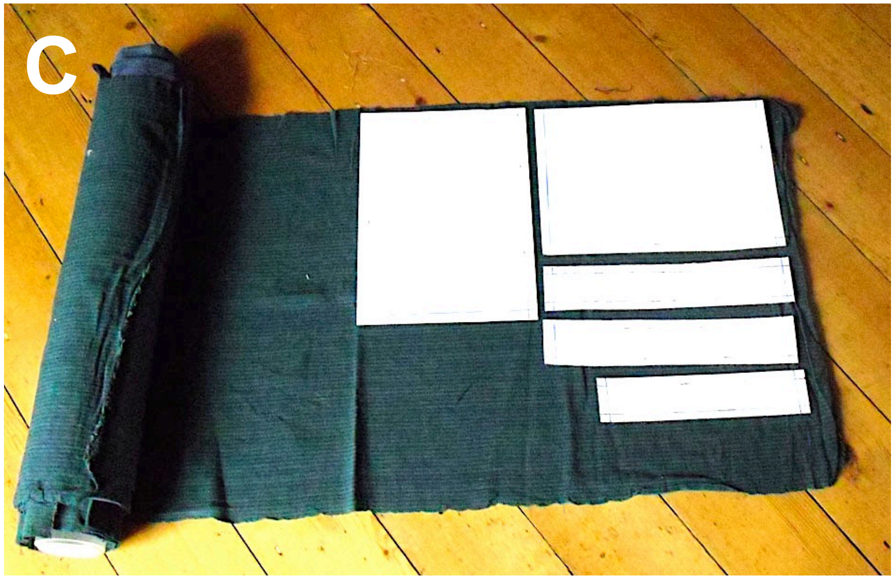


A Bolt of Fabric

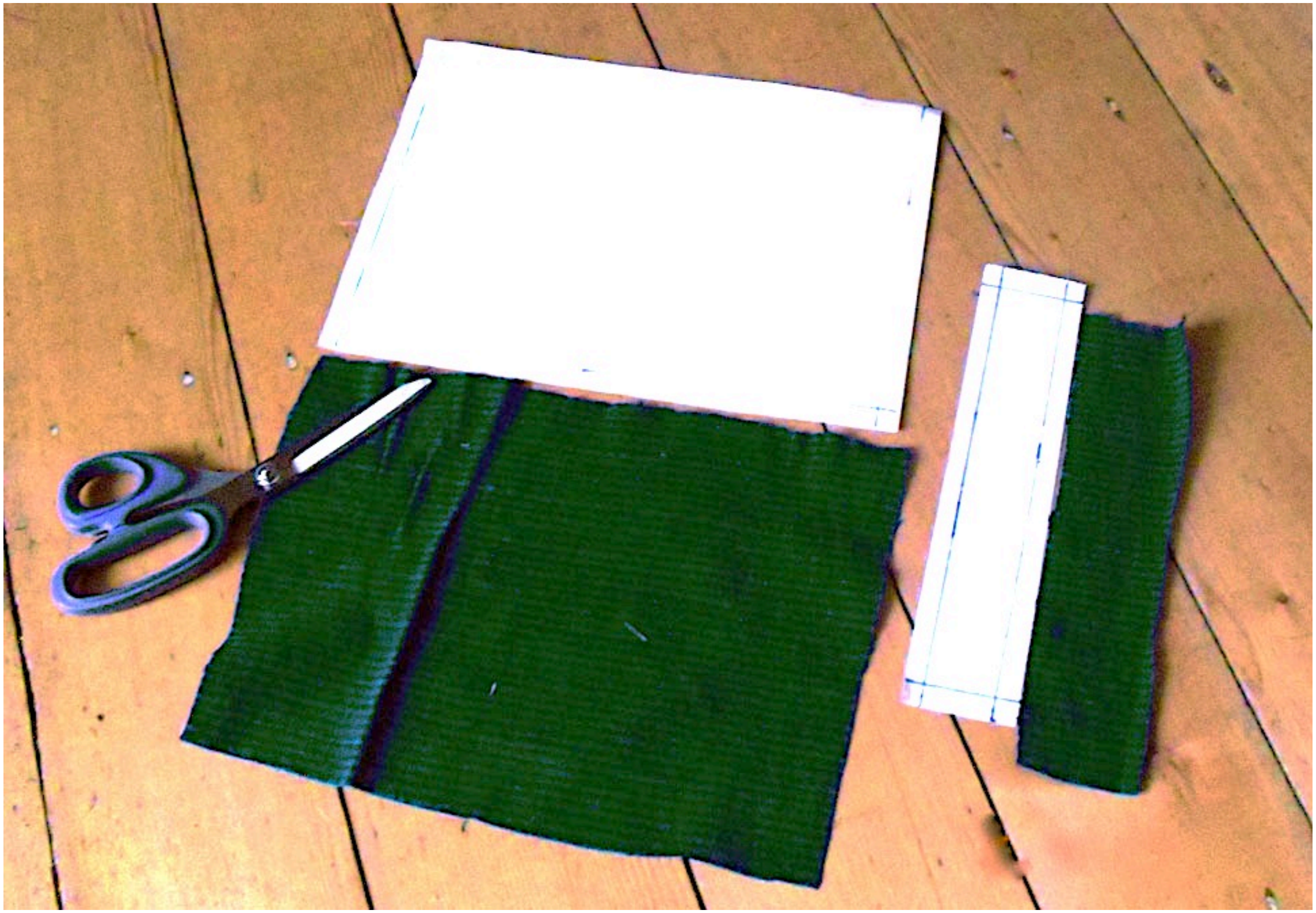
Arranging Pattern Pieces on Fabric



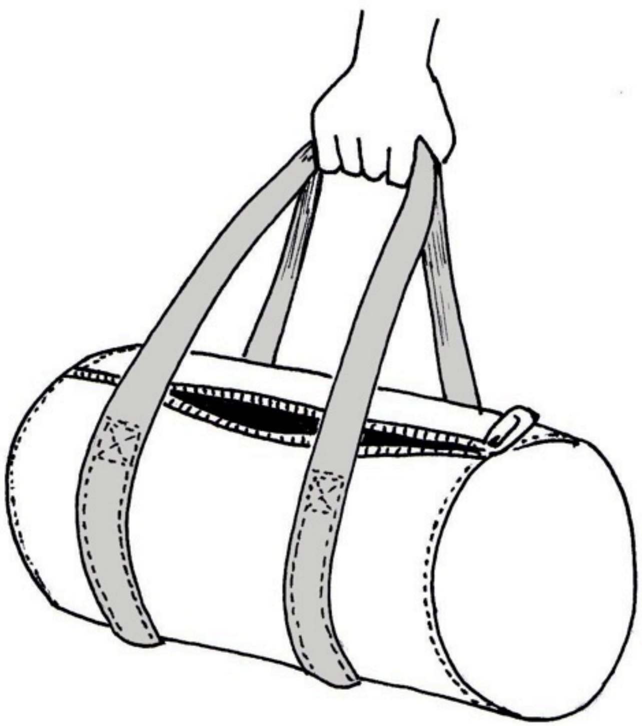
Which arrangement uses the least length of fabric?



Cutting the Pieces of Fabric



Designing a Sports Bag



- The length of the bag will be 20 inches.
- The bag will have circular ends of diameter 11 inches.
- The main body of the bag will be made from three pieces of fabric: a piece for the curved body, and the two circular end pieces.
- Each piece will need an extra ½ inch all the way around it so that the pieces can be stitched together. This is the seam allowance.

Analyzing Sample Responses to Discuss

(1) What method has the student used?

(2) What mistakes have been made?

(3) Has the student made assumptions and if so, what are they?

(4) How could the student improve their work?

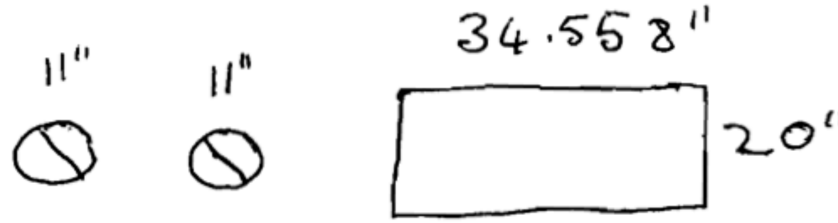
(5) How has looking at the student work helped you with your own solution to the problem?

Sample Responses to Discuss: Ben

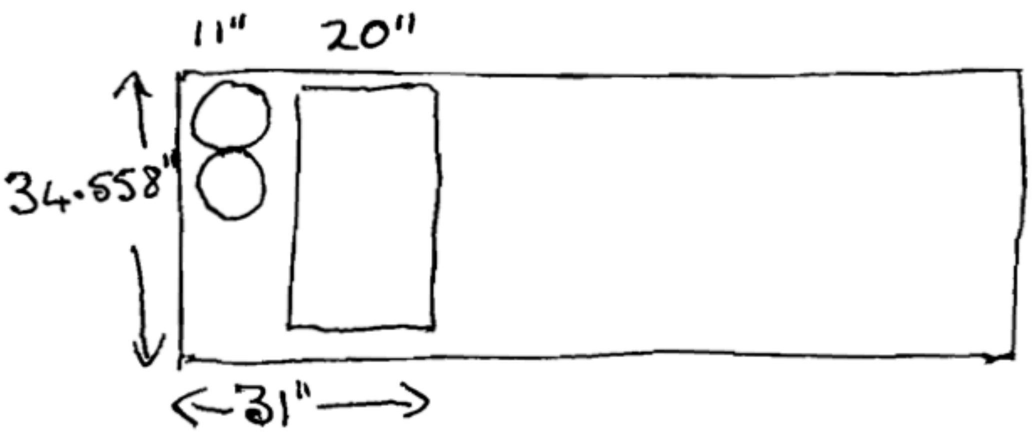
$c = 2\pi r$

$= 2 \times \pi \times 5.5$

$= 34.558$



3 pieces to cut



Sample Responses to Discuss: Aisha

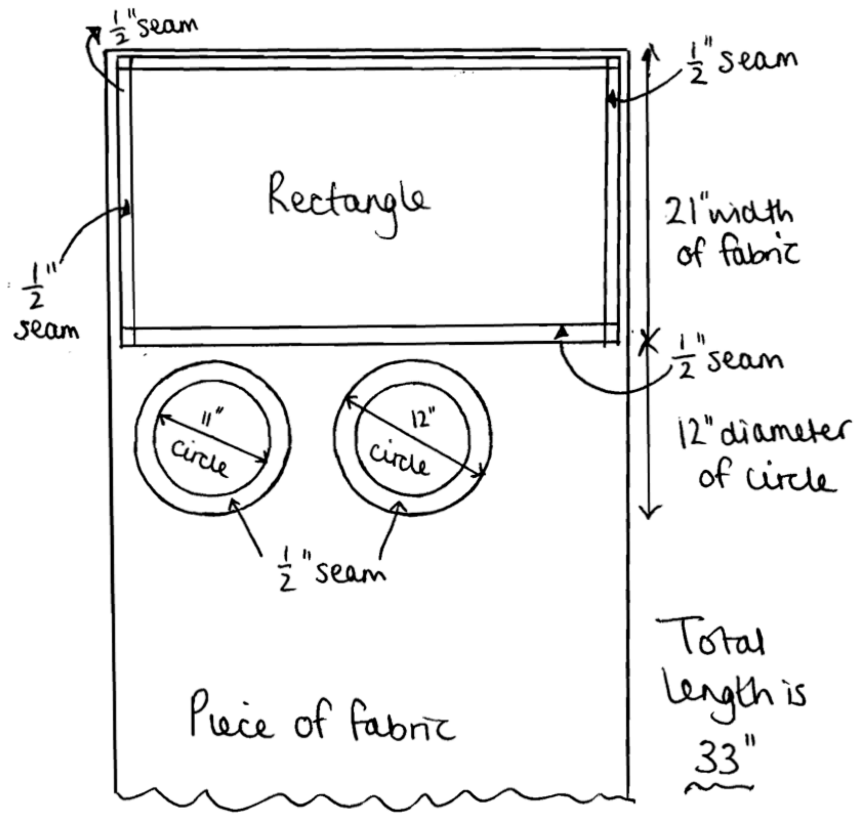
Scale 1:8

$C = \pi d = 3 \times 11$ ← This is the length of the rectangle

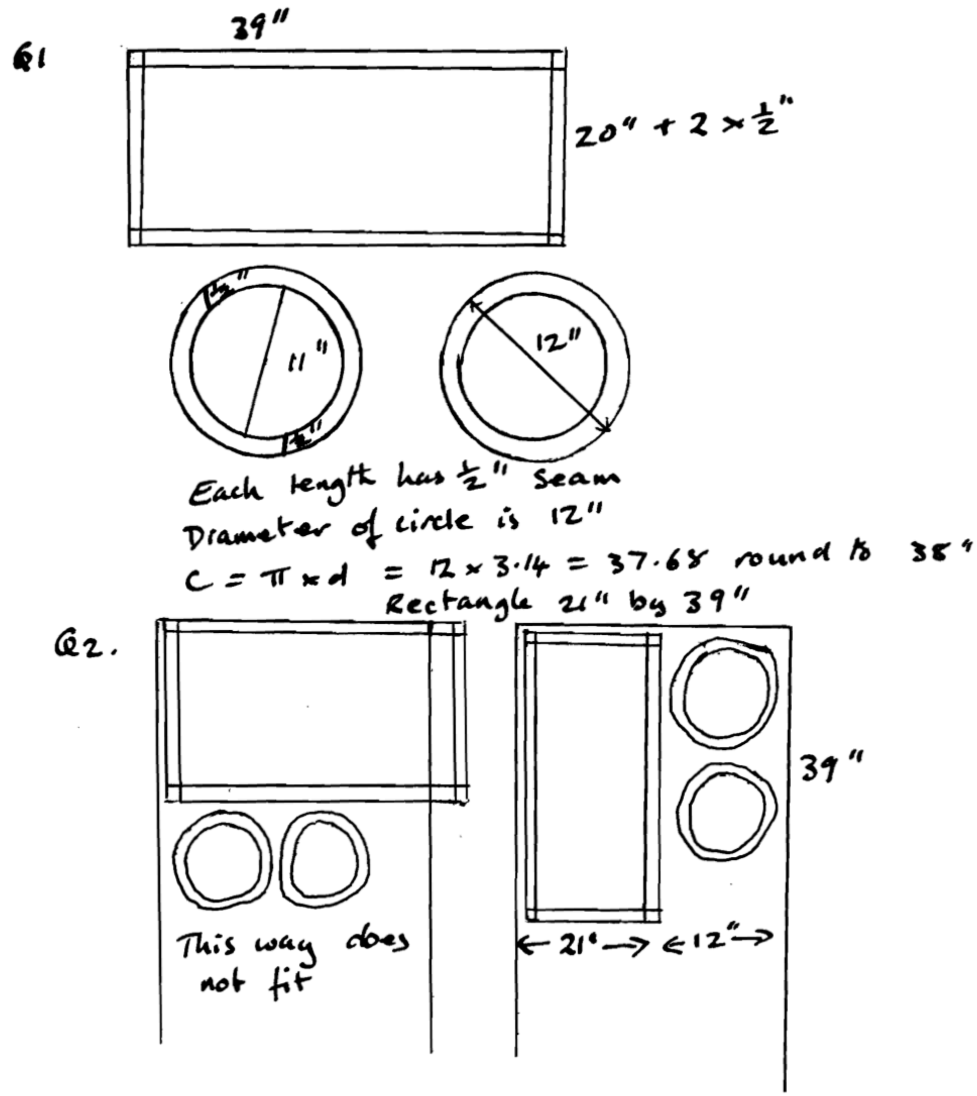
$C = 33''$

Add seams: 34" long rectangle.

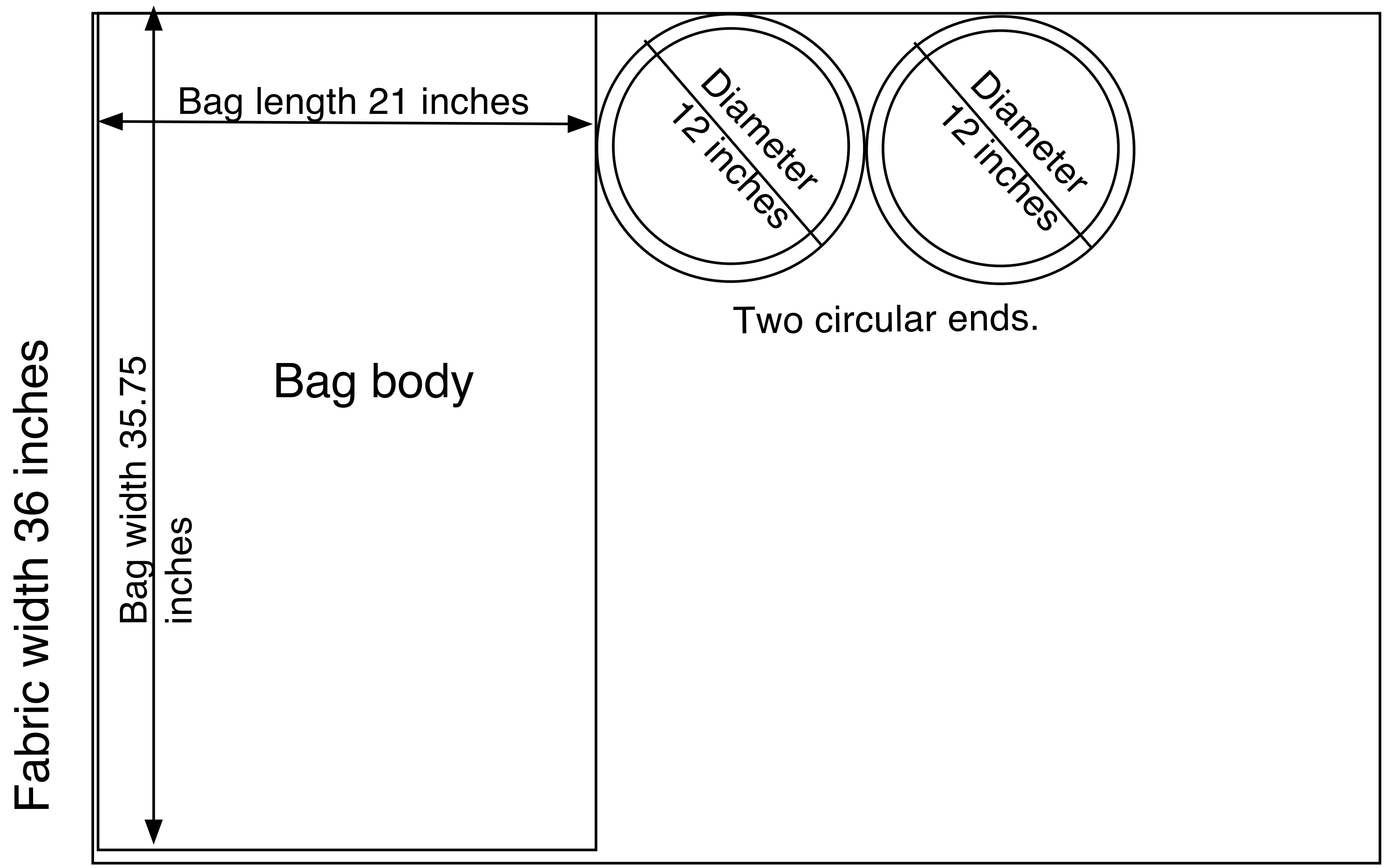
The fabric is 36" wide.



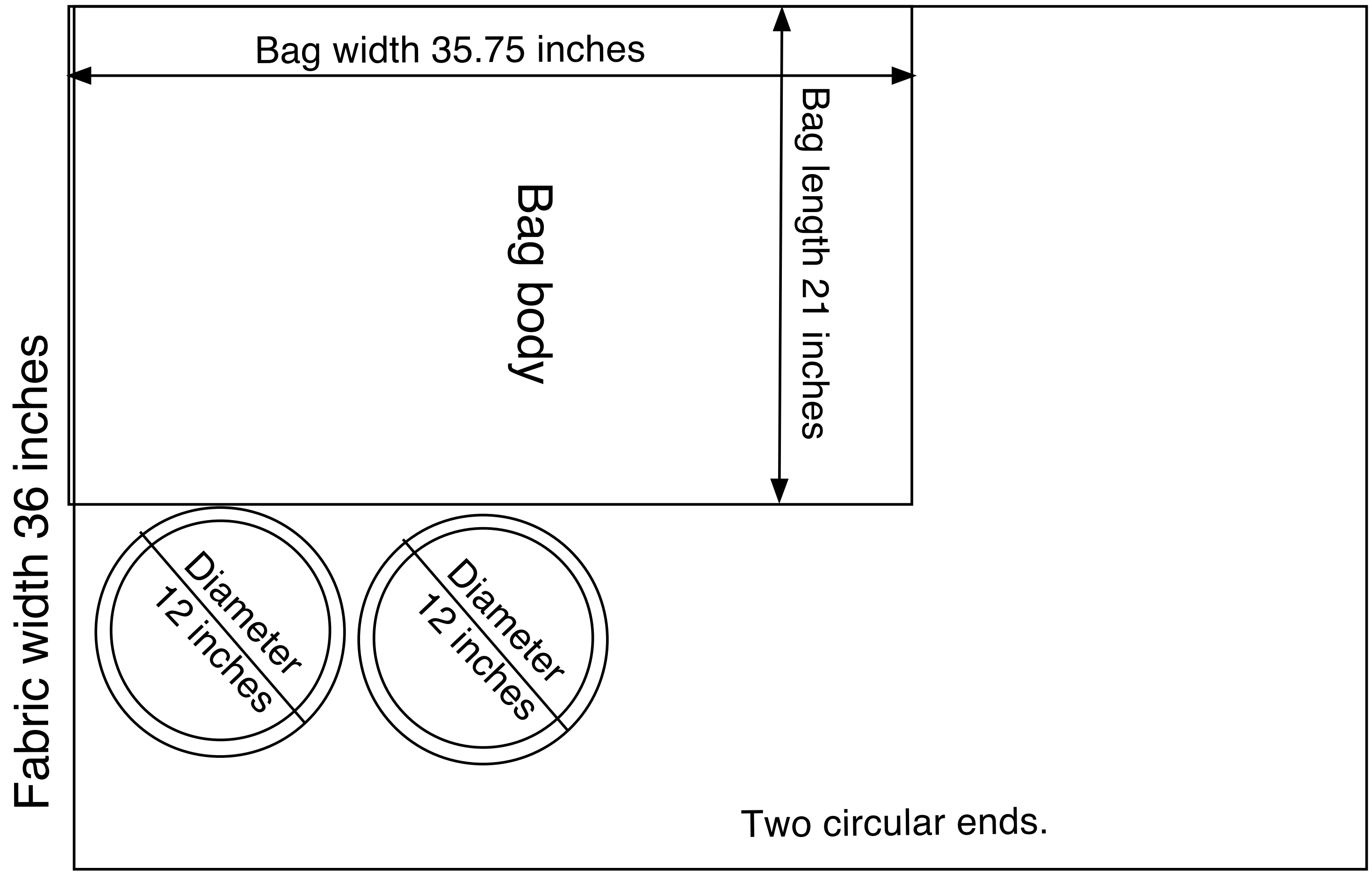
Sample Responses to Discuss: Carlotta



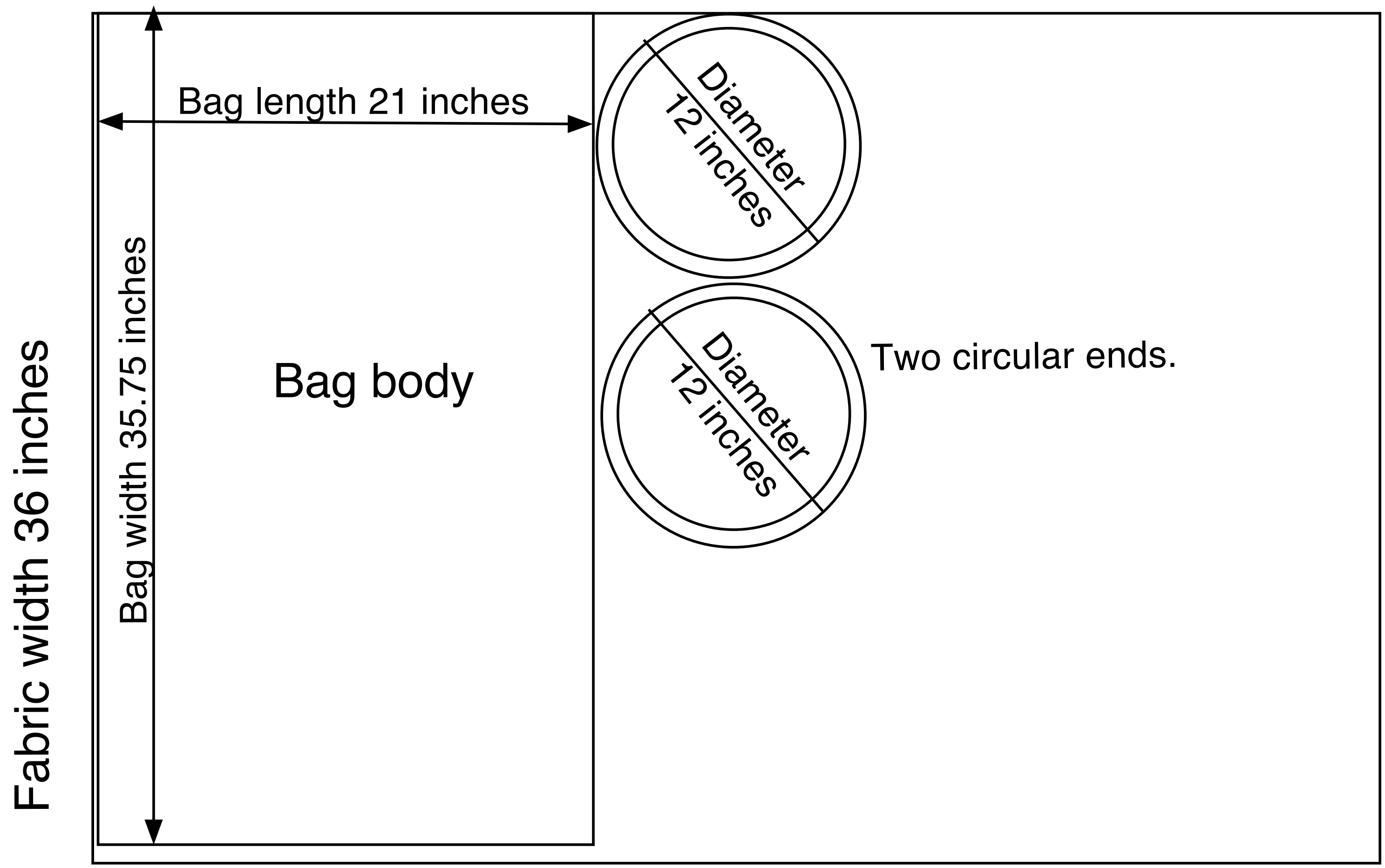
Which Arrangement Uses Least Length? (1)



Which Arrangement Uses Least Length? (2)



Which Arrangement Uses Least Length? (3)



Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
Shell Center Team at the Center for Research in Mathematical Education
University of Nottingham, England:

Malcolm Swan,
Nichola Clarke, Clare Dawson, Sheila Evans, Colin Foster, and **Marie Joubert**
with
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The central task in this lesson was originally designed for
Bowland Maths (http://www.bowlandmaths.org.uk)
and appears courtesy of the **Bowland Charitable Trust**

We are grateful to the many teachers and students, in the UK and the US,
who took part in the classroom trials that played a critical role in developing these materials

The classroom observation teams in the US were led by
David Foster, Mary Bouck, and **Diane Schaefer**

This project was conceived and directed for
The Mathematics Assessment Resource Service (MARS) by
Alan Schoenfeld at the University of California, Berkeley, and
Hugh Burkhardt, Daniel Pead, and **Malcolm Swan** at the University of Nottingham

Thanks also to Mat Crosier, Anne Floyde, Michael Galan, Judith Mills, Nick Orchard, and Alvaro Villanueva who contributed to the design and production of these materials

This development would not have been possible without the support of
Bill & Melinda Gates Foundation

We are particularly grateful to
Carina Wong, Melissa Chabran, and **Jamie McKee**

The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org