PROBLEM SOLVING

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Using Space Efficiently: *Packing a Truck*

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Using Space Efficiently: *Packing a Truck*

MATHEMATICAL GOALS

This lesson unit is intended to help students to:

- Reason precisely and defend their conclusions.
- Use mathematics to model a scenario concerning volume.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Mathematical Practices* in the *Common Core State Standards for Mathematics*:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
8. Look for and express regularity in repeated reasoning.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

6.G: Solve real-world and mathematical problems involving area, surface area, and volume.

INTRODUCTION

The lesson unit is structured in the following way:

- Before the lesson, students attempt the *Packing It In* task individually. You review their responses and formulate questions that will help them improve their work.
- At the start of the lesson, students read your comments and consider ways to improve their work.
- In pairs or threes, students work together to develop a better solution, producing a poster to show their conclusions and their reasoning.
- Then, in the same small groups, students look at some sample student work showing different approaches to the problem. They evaluate the strategies used and seek to improve the arguments given.
- In a whole-class discussion, students compare different solution methods.
- Finally, students reflect individually on their learning.

MATERIALS REQUIRED

- Each student will need a copy of *Packing it In* and *How Did You Work?* and some blank paper to work on. Provide calculators for students who wish to use them.
- Each pair of students will need the three *Sample Responses to Discuss*, a piece of poster paper, a marker and some glue.
- Have graph paper and if possible, unifix cubes or small cardboard boxes available.
- There is a projector resource to support whole-class discussions.

TIME NEEDED

20 minutes before the lesson, a 95-minute lesson (or two 50-minute lessons) and 10 minutes in a follow-up lesson. Timings given are approximate. Exact timings will depend on the needs of your class.

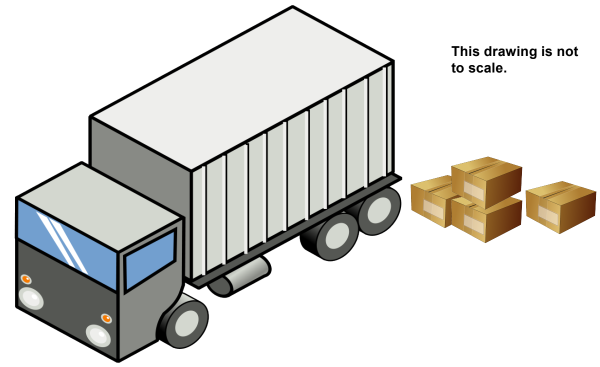
BEFORE THE LESSON

Introducing the task: *Packing It In* (20 minutes)

Ask the students to do this task, in class or for homework, a day or more before the lesson. This will give you an opportunity to assess their work and to find out the kinds of difficulties they have with it. You should then be able to target your help more effectively in the subsequent lesson.

Packing It In

Stefan is packing boxes into the back of a truck.



This drawing is not to scale.

The empty space in the back of the truck is 245 cm (W) by 250 cm (H) by 890 cm (L).

The boxes are all identical and measure 50 cm by 60 cm by 80 cm. They can be arranged in any way in the back of the truck.

Give instructions to Stefan on how to pack the truck so that the **maximum** number of boxes will fit in. State how many boxes will fit, if he packs the truck according to your instructions.

Give each student a copy of the task *Packing It In* and some blank paper to work on. Make sure that students understand the problem.

Has anyone ever had to load up a truck with boxes? When / Why?

Students might talk briefly about their experiences of moving house or helping an adult at work, for instance.

Read the task carefully and give some advice to Stefan about how to pack the truck. Make sure your instructions are clear and easy to follow.

Remember to state how many boxes will fit, if Stefan follows your packing instructions.

You can use a calculator if you wish.

It is important that, as far as possible, students are allowed to tackle the problem without assistance. If students are struggling to get started, then ask questions that help them understand what is required, but make sure you do not do the task for them.

Students who sit together often produce similar solutions so that, when they compare their work, they have little to discuss. For this reason we suggest that when students do the task individually you ask them to move to different seats. Then, at the beginning of the formative assessment lesson, allow them to return to their usual seats. Experience has shown that this may produce more profitable discussions.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding, and their problem solving strategies.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this unit. We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students at the beginning of the lesson.

Common issues:	Suggested questions and prompts:
Omits instructions For example: The student states that Stefan can get $20 \times 11 = 220$ boxes in the truck without any further explanation.	• Can you explain your calculation? • How exactly would Stefan need to pack the 220 boxes to get them in? • What instructions could you give to Stefan?
Does not take into account the box dimensions For example: The student divides the volume of the truck (54512500 cm^3) by the volume of a box (240000 cm^3) to give 227 boxes.	• Do the dimensions of the boxes affect how many will fit in the truck? • How exactly would you arrange the boxes to get this many in? Can you give some instructions to Stefan on how to pack the truck?
Specifies a non-integer value for the number of boxes For example: The student rounds the 245 cm measurement down to 240 cm and divides the volume of the truck by the volume of a box to give a total of 222.5 boxes.	• Is it possible to pack half a box? • How could you round your answer so that it makes sense in this context?
Inappropriate rounding For example: Students divide a dimension of the truck by a dimension of a box and round the answer up to give the number of boxes that will fit (e.g. $245 \div 50 = 4.9$ rounded up to 5.)	• What would the total length of the 5 boxes be? • What is the measurement of the side of the truck?
Does not fill the truck For example: The student considers just one layer of boxes rather than filling the whole truck.	• Is it possible to stack the boxes one on top of another? • How could you pack the truck to fill the space?
Orients the boxes all the same way For example: The student places the 50 cm sides along the 250 cm side, the 60 cm sides along the 245 cm side and the 80 cm sides along the 890 cm side.	• Do you think you could get more boxes in by placing some of them a different way around?
Considers only one way of packing For example: The student states that Stefan can get 220 boxes in the truck.	• How do you know that there is not some way of getting more boxes in? • How could you check that this is the maximum number of boxes that will fit in the truck?

SUGGESTED LESSON OUTLINE

Individual work (10 minutes)

Return the students' initial attempts at the *Packing It In* task. If you have not added questions to students' work, write a short list of your most common questions on the board. Students can then select a few questions appropriate to their own work.

Recall the work you did on the 'Packing It In' problem?

I have looked at your work and have some questions for you.

I would like you to think, on your own, about my questions and how your work could be improved.

Students may want to jot down their ideas as they consider how to improve their work. They can write directly on their original work using a different colored pen or could use a blank piece of paper.

Collaborative small-group work: joint solution (30 minutes)

Organize students into pairs or threes. You could deliberately put together students who have taken different approaches to the problem.

Slide P-2 of the projector resource summarizes how students should work together:

Collaborative Work

1. Take turns to explain the work that you have done so far.
2. Now try to come up with a joint solution that is better than the solutions that either of you produced individually.
3. Make a poster showing your solution, writing your reasoning and decisions in detail.

While students work in small groups, you have two tasks: to note different student approaches to the task and to support student problem solving.

Note different student approaches to the task

Listen to and watch students carefully and note their different approaches to the task. Do they make sketches of where the boxes will go or do they work purely numerically? Do they think about orienting the boxes so that they fit exactly along a side of the truck? Do they assume that all the boxes must be arranged the same way? Have they explained their conclusions clearly? This information will help you to focus a whole-class discussion later in the lesson.

Support student problem solving

Try not to make suggestions that move students towards a particular approach. Instead, ask questions that help them to clarify their thinking. The questions in the *Common issues* table may be helpful.

If students are struggling to get started, you could encourage them to choose an orientation for the boxes and see how many they can get in, not worrying too much about the wasted space. Even if they know that their initial attempt is not very efficient, it will help them get started. Then they can work to improve their answer.

Could you draw a diagram to help you?

Have you written down which way the boxes are going?

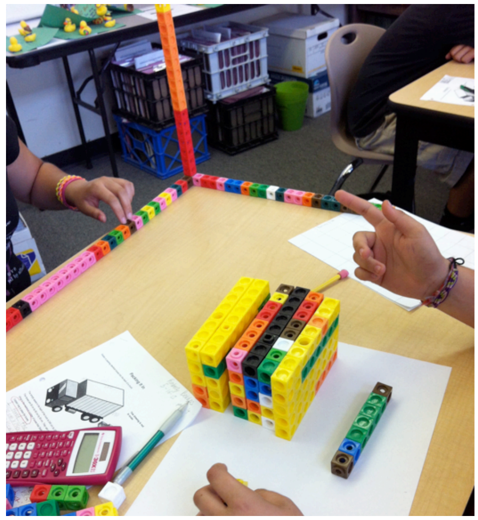
How many boxes are stacked along the side of the truck that is 245 cm long?

Would turning the boxes around change the number that you could fit in the truck?

How much space will be left after you have packed those boxes?

Could you fit another box in that space? Why / Why not?

In some trials, students have used unifix cubes to help them model the problem:



You may find it beneficial to have some unifix cubes or cardboard boxes available in the lesson to enable students to visualize the 3-dimensional context of the problem. If only unifix cubes are available, students could construct boxes from these.

Sharing posters (10 minutes)

Once students have had a chance to complete their joint solution to the task, ask one student from each group to visit the work of another group.

If you are staying with your poster, explain how you have packed the truck and how many boxes this allows Stefan to fit in.

If you are the visitor, look carefully at the work and listen to the explanation. Compare it with your own arrangement.

Visiting another group's poster allows students to share their thinking and provides a check, as to whether or not the instructions produced can be followed easily by someone else.

Extending the lesson over two days

If you are taking two days to complete the unit then you may want to end the first lesson here. At the start of the second day, briefly remind students of the problem before moving on to the collaborative analysis of sample responses.

Collaborative analysis of *Sample Responses to Discuss* (30 minutes)

Distribute copies of the *Sample Responses to Discuss* to each pair of students along with some blank paper. In trials of this lesson unit many students have attempted the problem by dividing the volume of the truck by the volume of a box to give the maximum possible number of boxes, without considering the dimensions of the boxes and how they can be packed into the truck. The sample student work has been designed to present a variety of *different* possible approaches to the task for students to consider, none of which provide a complete solution.

There may not be time, and it is not essential, for all groups to look at all three sample responses, so you may want to be selective about what you hand out. For example, groups that have successfully used one method may benefit from looking at a different approach. Other groups that have struggled with a particular approach may gain something from seeing a student version of the same kind of strategy. For example, groups that have not considered having boxes along the same side oriented in different directions might learn something from Moses' work.

In your groups you are now going to look at some student work on the task.

Notice in what ways this work is similar to yours and in what ways it is different.

There are some questions for you to answer as you look at the work.

You may want to add notes to the work to make it easier to follow.

Slide P-3 of the projector resource describes how students should work together:

Evaluating Sample Student Responses

1. Look carefully through each student's solution.
2. Talk about the methods they have used.
3. Answer the questions on the sheet. Use blank paper if you need more space.
4. Discuss what is good and bad about each solution and whether there are any ideas that you can use.

Encourage students to focus on the approaches and math used in the sample work, rather than superficial features, such as whether the student has neat handwriting. The arithmetic in the sample responses is correct and so students do not need to be concerned with looking for mistakes of this type. Instead, students should be encouraged to evaluate the strategies used, identifying errors in rounding (Leillah), missing details about how to pack the truck (Faridah) and lack of optimization (Moses).

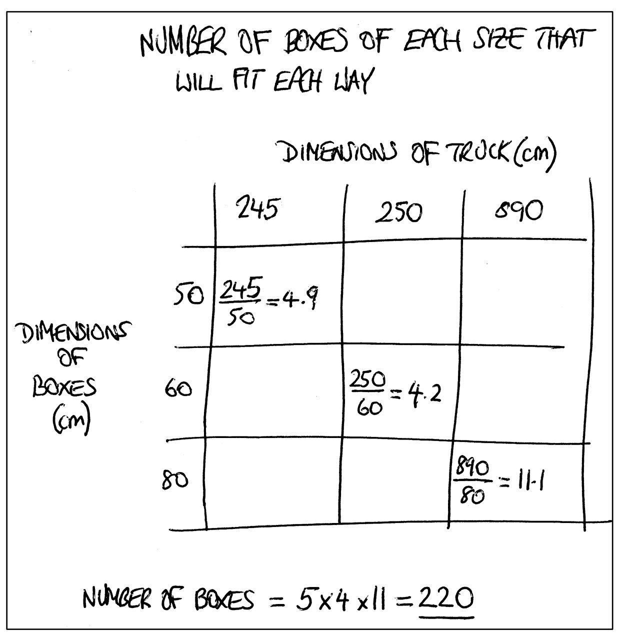
During the small group work, support the students as in the first collaborative activity. Also, check to see which of the responses students find more difficult to understand. Note similarities and differences between the sample approaches and those the students took in the collaborative work.

Leillah has begun to make a table to show how many times each box dimension will fit into each truck space dimension.

The issue of what to do with non-integer answers has arisen and she seems to be using normal rounding conventions. Her answer does not make sense, as 5 of the 50 cm sides of the boxes cannot fit into the 245 cm length on the truck. Leillah needs to round **down** her division answers and also to consider the other six calculations that she has omitted in order to find the best combination that maximizes the product of the three values.

How many 50 cm sides of the box will fit along the 245 cm side of the truck?

Would these sides of the boxes fit better



down another side of the truck? How could Leillah check?

Where do the figures in Leillah's multiplication come from? Do they make sense?

Leillah's work is just the beginning of a complete solution. You could suggest students correct and complete Leillah's work.

I will work in decimeters (dm)

1dm = 10 cm

Truck is 24·5 × 25 × 89 . Boxes are 5 × 6 × 8

extra 0·5 cannot be filled so will consider 24 × 25 × 89

Factors are:

24 : 1, 2, 3, 4, 6, 8, 12, 24

25 : 1, 5, 25

89 : 1, 89

Put the 6dm and the 8dm sides on the 24dm (24·5 dm) side

Put the 5dm side on the 25dm side of the truck so no wasted space

Faridah sensibly chooses to work in dm rather than cm. Not only does this make the numbers smaller and more manageable, but also this enables her to think more easily about the divisors (factors) of the truck's dimensions.

Why does Faridah say that the extra 0.5 dm cannot be filled?

Why does Faridah figure out the factors of numbers?

Her answer suggests that placing the 5 dm box sides along the 25 dm side of the truck will lead to efficient packing, but she has not resolved what to do about the other two sides of the truck.

What decision does Faridah need to make about how to pack the truck?

In particular, the fact that 89 is prime means that no consistent orientation of the boxes will utilize all of the available space in that direction. However, this is not necessarily the case if the boxes do not all have to have the same orientation, as Moses finds.

Moses discovers that he can place boxes along the 890 cm side of the truck by using 4 boxes with their 60 cm sides and 13 boxes with their 50 cm sides. He uses a sketch to illustrate which way the boxes go and to help him consider which lengths will lie along the other sides.

What does the box labeled '13 × 50 cm' represent?

He does not show exactly where subsequent boxes will go. His instruction to 'repeat this pattern' is not precise enough to determine exactly what he has in mind.

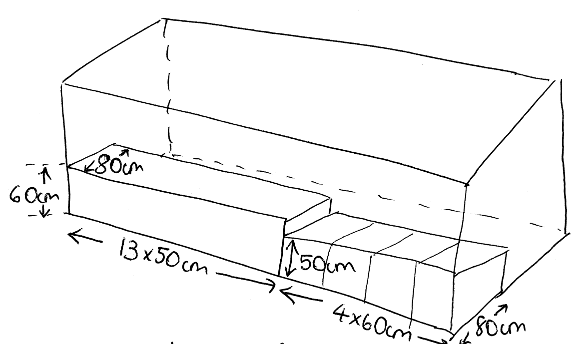
How will repeating this arrangement of 17 boxes fill the truck?

4 x 60cm = 240cm

13 x 50cm = 650cm

890cm

4+13 = 17 boxes altogether



no wasted space along the longest side

Then repeat this pattern enough times to fill the rest of the space.

If the strip created is repeated twice more along the floor of the truck (17 x 3 = 51 boxes), and this layer is repeated three times more above, that would lead to 4 x 51 = 204 boxes in total. The gaps remaining will not accommodate any more boxes, but a different arrangement could make this space useable for further boxes.

Whole-class discussion: comparing different solution methods (15 minutes)

Hold a whole-class discussion to consider the different approaches used by members of the class and in the sample work. Ask the students to compare the different solution methods.

What approach did you take?

Can you describe the three sample students' approaches?

What were the advantages and disadvantages of the different approaches?

Which approach did you think was the most effective? Why?

Which approach did you find most difficult to understand? Why?

How could each student improve his or her solution?

Did anyone use a similar approach to one of the sample students' approaches? How was it similar? How was it different?

Try to focus the discussion on any common misconceptions you noticed in the collaborative work. You may want to draw on the questions in the *Common issues* table to support your own questioning.

The discussion may draw on aspects of modeling real-life situations, such as whether the boxes will really all be exactly the same size, and whether, for example, five 50 cm lengths will fit into 250 cm when it is such a tight fit. These questions could be related to measurement errors, where a real-life measurement of 250 cm might in certain circumstances be taken to be between 249.5 cm and 250.5 cm, or perhaps 245 cm and 255 cm, etc. Another issue that students might be concerned about is whether the *weight* of all the boxes, depending on what they might contain, is reasonable for the truck.

You may want to use Slides P-4 to P-6, which show the sample student responses.

Follow-up lesson: individual reflection (10 minutes)

Give each student a copy of the questionnaire *How Did You Work?* The questionnaire should help students to review their progress.

Think carefully about your work on this task and the different methods you have seen and used.

On your own, answer the review questions as carefully as you can.

If you have time, you may also want to ask your students to re-read their original solutions to *Packing It In* and using what they have learned, attempt the task again, perhaps using a different method. In this case, give each student a blank copy of the task *Packing It In*.

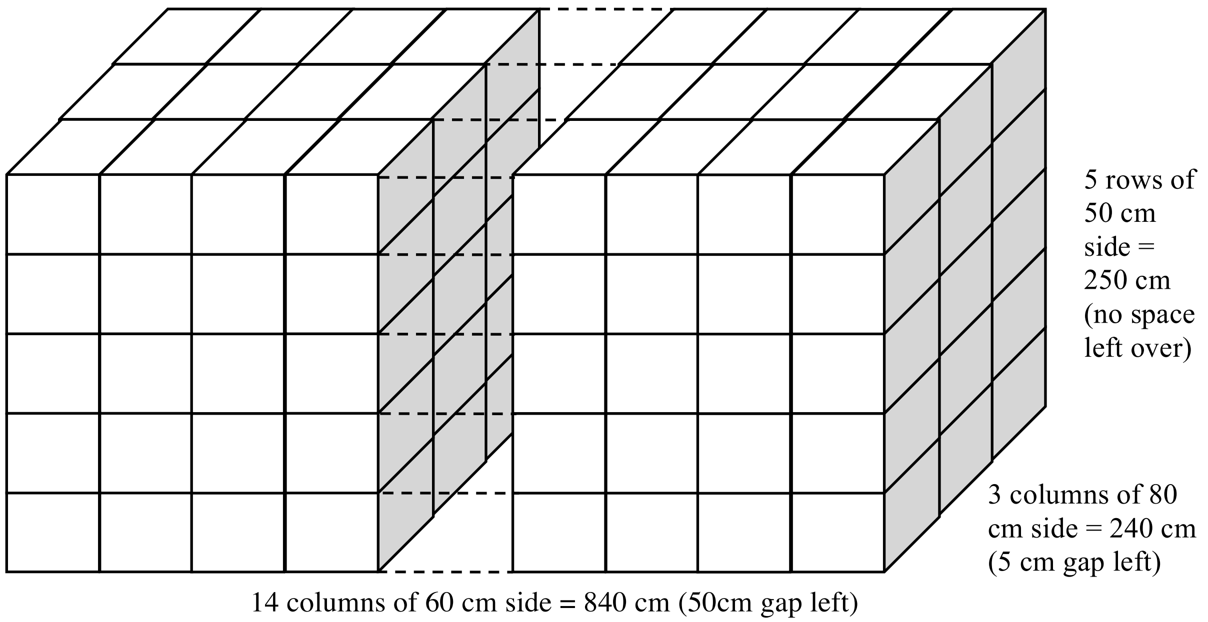
Some teachers give this task for homework.

SOLUTIONS

Packing It In

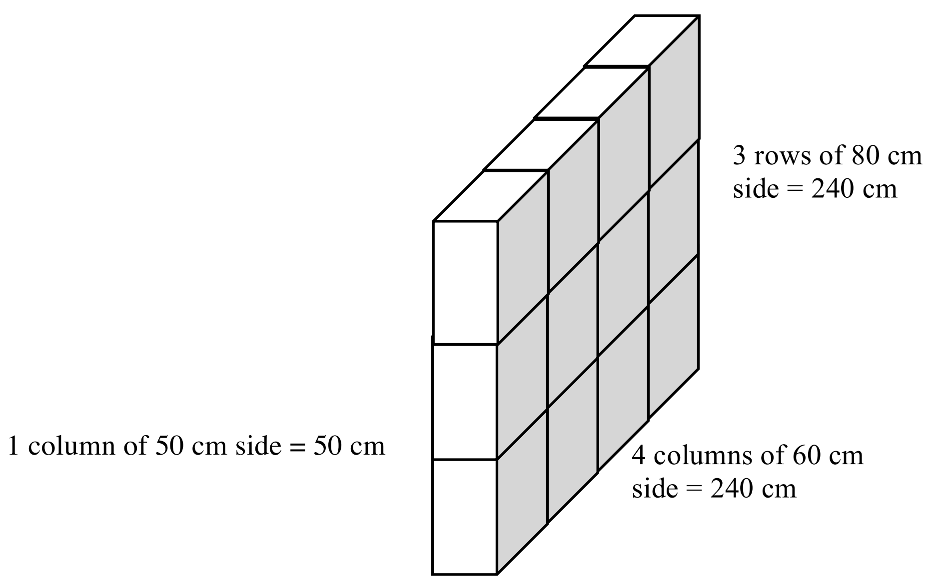
There are many sensible ways of approaching this problem. Dividing the volume of the truck by the volume of a box offers a good starting point, as it gives an upper bound for the maximum possible number of boxes that might conceivably fit. However, this calculation leads to 227 boxes, whereas it is only possible to fit 222 boxes in the truck. A possible arrangement of the boxes is illustrated below.

Firstly, 210 boxes are placed as shown:

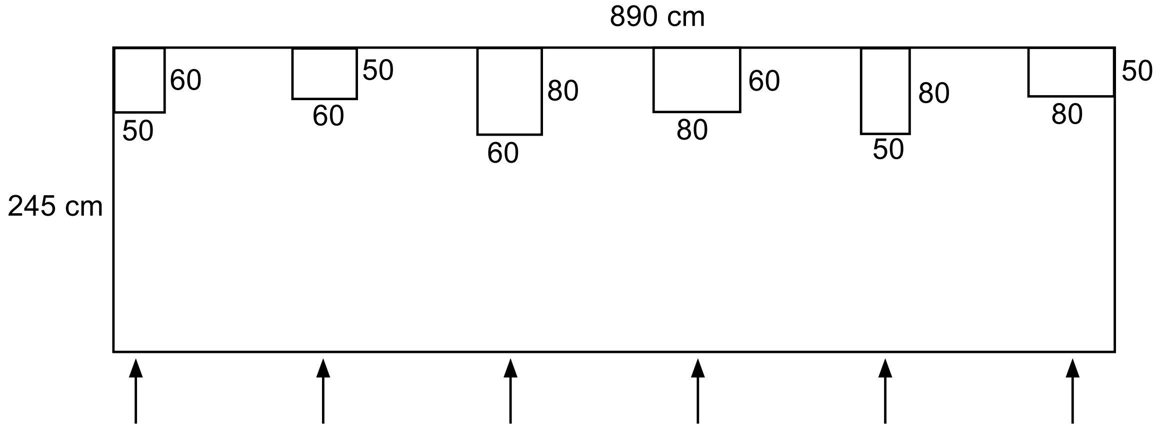


This arrangement contains 14 x 3 x 5 = 210 boxes.

This leaves a 5 cm gap, which cannot be filled and a space at the end of depth 50 cm, which is not enough to accommodate another row of boxes this way round. However, if the orientation of additional boxes are **turned around**, then an extra column of 12 boxes will fit, making 222 boxes altogether:



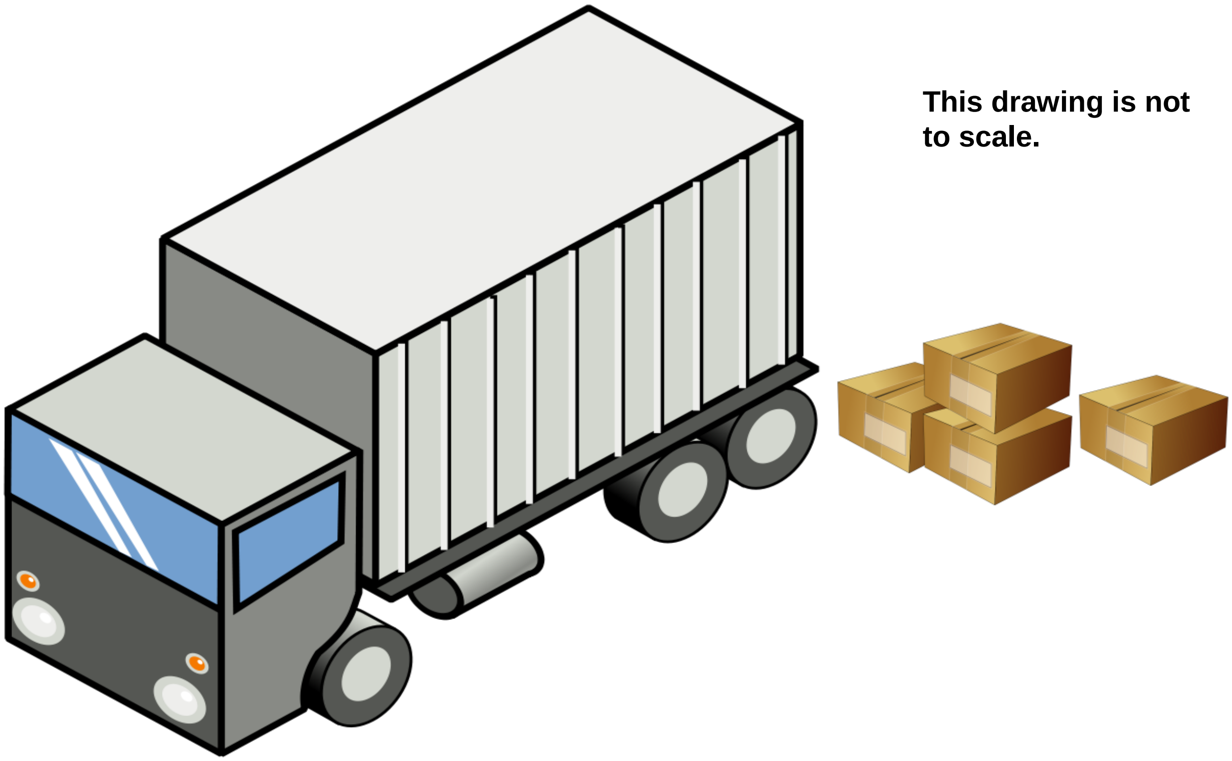
The diagram below shows all six possible orientations when all the boxes are placed in the same direction:



245cm	4	4	3	4	3	4
	×	×	×	×	×	×
890 cm	17	14	14	11	17	11
	×	×	×	×	×	×
250 cm	3	3	5	5	4	4
Boxes	**204**	**168**	**210**	**220**	**204**	**176**

Packing It In

Stefan is packing boxes into the back of a truck.



This drawing is not to scale.

The empty space in the back of the truck is 245 cm (W) by 250 cm (H) by 890 cm (L).

The boxes are all identical and measure 50 cm by 60 cm by 80 cm. They can be arranged in any way in the back of the truck.

Give instructions to Stefan on how to pack the truck so that the **maximum** number of boxes will fit in. State how many boxes will fit, if he packs the truck according to your instructions.

Sample Responses to Discuss: Leillah

NUMBER OF BOXES OF EACH SIZE THAT WILL FIT EACH WAY

DIMENSIONS OF TRUCK (cm)

DIMENSIONS OF BOXES (cm)	245	250	890
50	$\frac{245}{50} = 4.9$		
60		$\frac{250}{60} = 4.2$	
80			$\frac{890}{80} = 11.1$

NUMBER OF BOXES $= 5 \times 4 \times 11 = \underline{220}$

1. Try to explain what Leillah has begun to do.

2. Do you agree with her conclusion? Why / Why not?

Sample Responses to Discuss: Faridah

I will work in decimeters (dm)

1 dm = 10 cm

Truck is 24·5 × 25 × 89. Boxes are 5 × 6 × 8

↑ extra 0·5 cannot be filled so will consider 24 × 25 × 89

Factors are:

24 : 1, 2, 3, 4, 6, 8, 12, 24

25 : 1, 5, 25

89 : 1, 89

Put the 5dm side on the 25dm side of the truck so no wasted space

Put the 6dm and the 8dm sides on the 24dm (24·5 dm) side

1. Try to explain what Faridah has done.

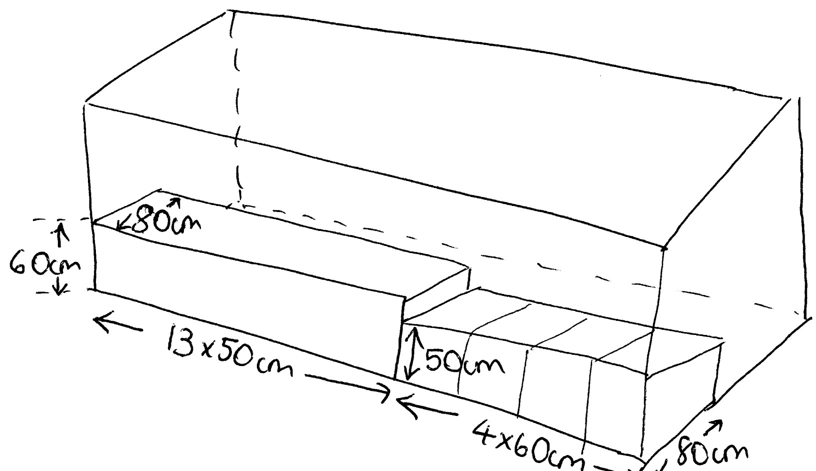
2. What are the advantages of using decimeters rather than centimeters?

3. Do you agree with her reasoning? Why / Why not?

Sample Responses to Discuss: Moses

$4 \times 60\text{cm} = 240\text{cm}$
$13 \times 50\text{cm} = 650\text{cm}$
890cm

$4 + 13 = 17$ boxes altogether



no wasted space along the longest side

Then repeat this pattern enough times to fill the rest of the space.

1. Try to explain what Moses has begun to do.

2. What are the strengths and weaknesses of his approach?

How Did You Work?

Complete the boxes and sentences that apply to your work:

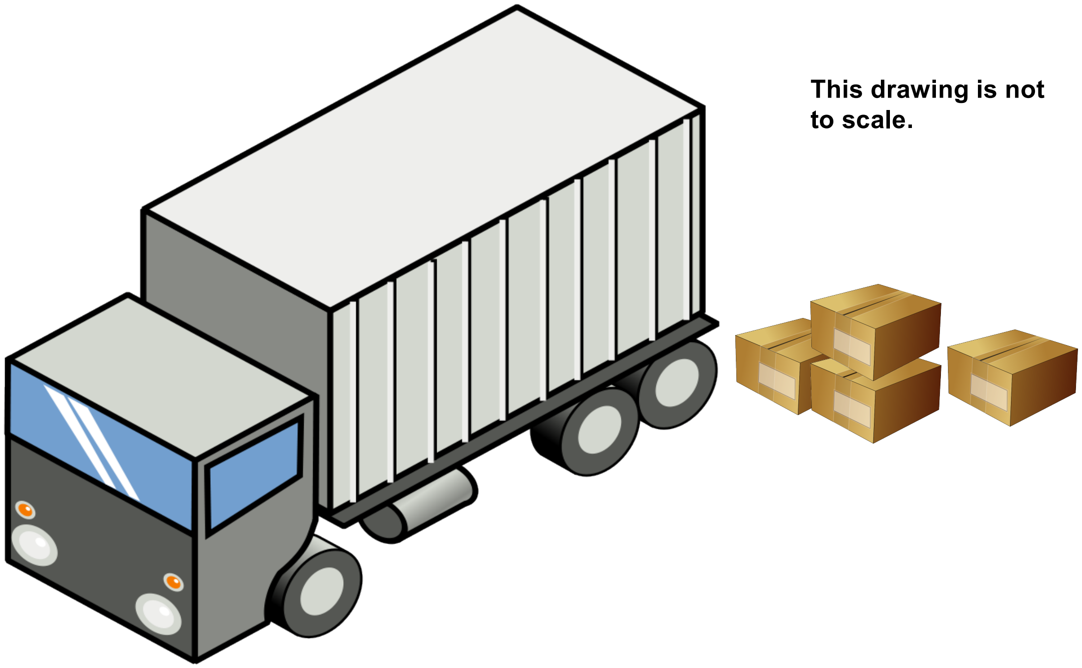
1. By following my own instructions the number of boxes Stefan could fit in the truck was: ☐ boxes.
2. By following our group's instructions the number of boxes Stefan could fit in the truck was: ☐ boxes.
3. When producing a joint solution we improved our work by: ....
4. We checked we had found the maximum number of boxes ☐ **OR** We could have checked by: ....

   We checked by: ....

5. Looking at the sample student work was helpful because: ....
6. My advice to a student new to this task would be: ....

Packing It In

Stefan is packing boxes into the back of a truck.



The empty space in the back of the truck is 245 cm (W) by 250 cm (H) by 890 cm (L).

The boxes are all identical and measure 50 cm by 60 cm by 80 cm.

They can be arranged in any way in the back of the truck.

Collaborative Work

1. Take turns to explain the work that you have done so far.

2. Now try to come up with a joint solution that is better than the solutions that either of you produced individually.

3. Make a poster showing your solution, writing your reasoning and decisions in detail.

Evaluating Sample Student Responses

1. Look carefully through each student's solution.
2. Talk about the methods they have used.
3. Answer the questions on the sheet. Use blank paper if you need more space.
4. Discuss what is good and bad about each solution and whether there are any ideas that you can use.

Sample Responses to Discuss: Leillah

NUMBER OF BOXES OF EACH SIZE THAT WILL FIT EACH WAY

		DIMENSIONS OF TRUCK (cm)		
		245	250	890
DIMENSIONS OF BOXES (cm)	50	$\frac{245}{50} = 4.9$		
	60		$\frac{250}{60} = 4.2$	
	80			$\frac{890}{80} = 11.1$

NUMBER OF BOXES $= 5 \times 4 \times 11 = \underline{220}$

Sample Responses to Discuss: Faridah

I will work in decimeters (dm)

1dm = 10cm

Truck is 24·5 × 25 × 89. Boxes are 5 × 6 × 8

extra 0·5 cannot be filled so will consider 24 × 25 × 89

Factors are:

24 : 1, 2, 3, 4, 6, 8, 12, 24

25 : 1, 5, 25

89 : 1, 89

Put the 5dm side on the 25dm side of the truck so no wasted space

Put the 6dm and the 8dm sides on the 24dm (24·5dm) side

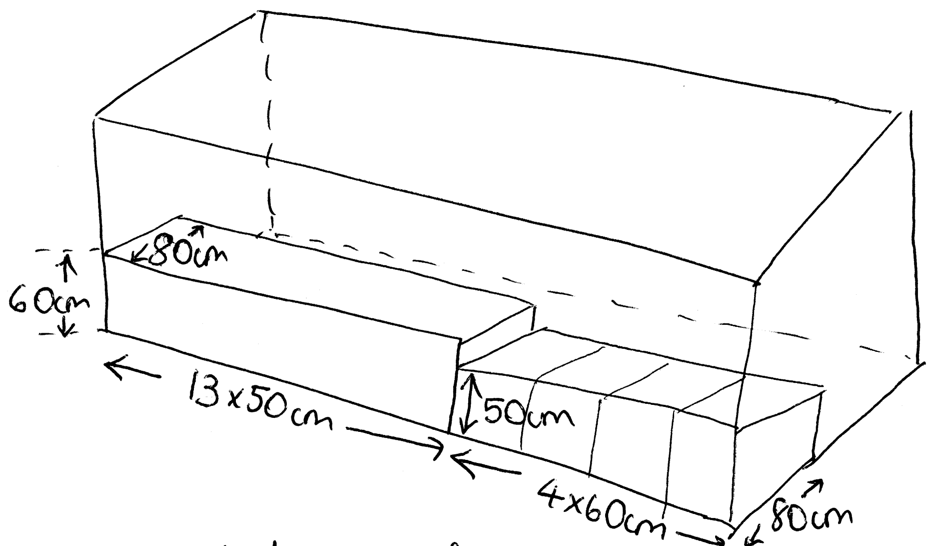
Sample Responses to Discuss: Moses

$4 \times 60cm = 240cm$

$13 \times 50cm = 650cm$

$\underline{890cm}$

$4 + 13 = 17$ boxes altogether



no wasted space along the longest side

Then repeat this pattern enough times to fill the rest of the space.

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
Shell Center Team at the Center for Research in Mathematical Education
University of Nottingham, England:

Malcolm Swan,
Nichola Clarke, Clare Dawson, Sheila Evans, Colin Foster, and **Marie Joubert**
with
Hugh Burkhardt, Rita Crust, Andy Noyes, and **Daniel Pead**

We are grateful to the many teachers and students, in the UK and the US,
who took part in the classroom trials that played a critical role in developing these materials

The classroom observation teams in the US were led by
David Foster, Mary Bouck, and **Diane Schaefer**

This project was conceived and directed for
The Mathematics Assessment Resource Service (MARS) by
Alan Schoenfeld at the University of California, Berkeley, and
Hugh Burkhardt, Daniel Pead, and **Malcolm Swan** at the University of Nottingham

Thanks also to Mat Crosier, Anne Floyde, Michael Galan, Judith Mills, Nick Orchard, and Alvaro Villanueva who contributed to the design and production of these materials

This development would not have been possible without the support of
Bill & Melinda Gates Foundation

We are particularly grateful to
Carina Wong, Melissa Chabran, and **Jamie McKee**

The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org