CONCEPT DEVELOPMENT

Mathematics Assessment Project
CLASSROOM CHALLENGES
A Formative Assessment Lesson

Finding Factors and Multiples

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Finding Factors and Multiples

MATHEMATICAL GOALS

This lesson unit is intended to help you to assess how well students are able to understand the meanings of the terms 'Greatest Common Factor' (GCF) and 'Least Common Multiple' (LCM). In particular, the lesson will help you to identify and help students who:

- Confuse GCFs and LCMs.
- Lack a sense of what values of GCF or LCM might be reasonable for a given pair of numbers.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

6.NS: Compute fluently with multi-digit numbers and find common factors and multiples.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 3 and 7:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

This lesson is structured in the following way:

- Before the lesson, students work individually on an assessment task, *Thinking of Two Numbers*, which is designed to reveal their current understanding and difficulties. You then review their solutions and create questions for students to consider in order to improve their work.
- In a whole-class introduction to the lesson, students review their understanding of GCF and LCM. They then examine some sample student work and evaluate the different approaches taken.
- Students then take part in a collaborative task, in pairs or groups of three, in which they complete a grid, finding numbers to satisfy different constraints of GCF and LCM. As they do this, they learn more about how these quantities are related to each other and to the numbers.
- Then in a whole-class discussion, students consider their approaches and share their conclusions.
- In a follow-up lesson, students receive your comments on the assessment task and use these to attempt a similar task, *Thinking of Two Numbers (revisited)*, approaching it with insights that they have gained from the lesson.

MATERIALS REQUIRED

- Each student will need a copy of the assessment task *Thinking of Two Numbers* and *Thinking of Two Numbers (revisited),* a mini-whiteboard, pen, and eraser.
- Each small group of students will need cut-up *Card Set: Sets of Numbers*, a copy of the grid *GCF and LCM,* and a glue stick. (The *Card Set: Blank Cards* is available for students needing an extension.)

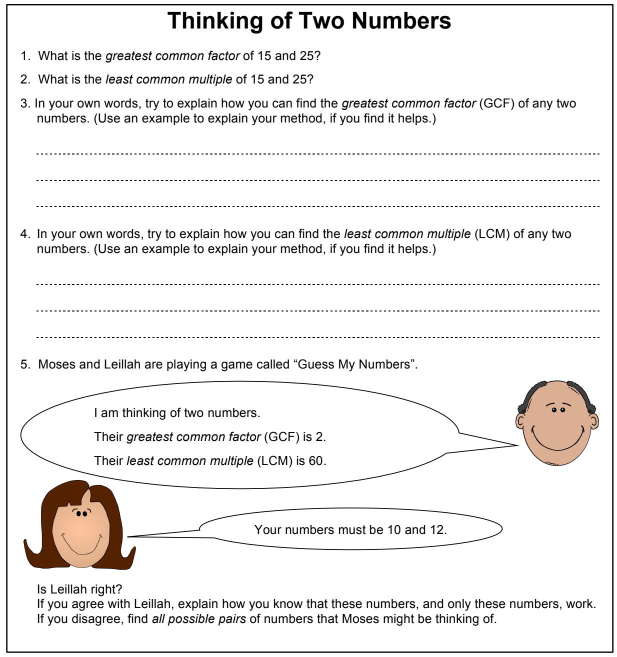
TIME NEEDED

20 minutes before the lesson, a one-hour lesson and 15 minutes in a follow-up lesson. Timings given are approximate and will depend on the needs of your class.

BEFORE THE LESSON

Assessment task: *Thinking of Two Numbers* (20 minutes)

Have the students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.


Thinking of Two Numbers

1. What is the *greatest common factor* of 15 and 25?
2. What is the *least common multiple* of 15 and 25?
3. In your own words, try to explain how you can find the *greatest common factor* (GCF) of any two numbers. (Use an example to explain your method, if you find it helps.)
4. In your own words, try to explain how you can find the *least common multiple* (LCM) of any two numbers. (Use an example to explain your method, if you find it helps.)
5. Moses and Leillah are playing a game called "Guess My Numbers".

I am thinking of two numbers.
Their *greatest common factor* (GCF) is 2.
Their *least common multiple* (LCM) is 60.

Your numbers must be 10 and 12.

Is Leillah right?
If you agree with Leillah, explain how you know that these numbers, and only these numbers, work.
If you disagree, find *all possible pairs* of numbers that Moses might be thinking of.

Give each student a copy of the task *Thinking of Two Numbers*.

Introduce the task briefly:

> *In this task you will need to use your knowledge of factors and multiples to answer the questions.*
>
> *Explain your work clearly so that I can understand your method.*

It is important that, as far as possible, students answer the questions on the sheet without assistance. If students are struggling to get started, ask questions that help them understand what they are being asked to do, but do not do the task for them.

Students should not worry too much if they cannot understand or do everything, because there will be a lesson related to this, which should help them. Explain to students that by the end of the next lesson they should expect to answer questions such as these confidently; this is their goal.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different approaches to the task.

We suggest that you do not score students' work. Research suggests that this will be counter-productive, as it will encourage students to compare their scores, and will distract their attention from what they can do to improve their mathematics. Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this unit.

We recommend that you:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these questions on the board when you return the work to the students in the follow-up lesson.

Common issues	Suggested questions and prompts
Has difficulty getting started	• What can you tell me about a 'factor'? • What do we mean by a 'multiple'?
Confuses the terms GCF and LCM	• What is the difference between the '**greatest** common factor' and the '**least** common multiple'?
Omits or has difficulty with providing an explanation For example: Student provides an example, but does not explain how to find the greatest common factor/least common multiple (Q3 and/or Q4).	• What did you do first? What did you do next? • How could you describe the process you use for finding the greatest common factor/least common multiple?
Thinks Leillah's numbers do not work For example: Student says that Moses' numbers **cannot** be 10 and 12 (Q5).	• What is the greatest common factor of 10 and 12? • What is the least common multiple of these numbers?
Assumes Moses' numbers must be 2 and 60 For example: Student points out that the greatest common factor of 2 and 60 is 2 and the least common multiple of 2 and 60 is 60 (Q5).	• You have found a pair of numbers that work. How could you check to see if there are others that also work?
Agrees with Leillah for an inadequate reason For example: Student points out that $2 \times 60 = 10 \times 12$ and appears to regard this as sufficient justification (Q5).	• Can you think of some more numbers that multiply to make 120? What would the greatest common factor and least common multiple of these numbers be?
Finds some correct possibilities for Moses' numbers, but not all four For example: Student finds (2, 60) and (6, 20) but not (4, 30) (Q5).	• How could you check that you have found all the possible answers?
Finds all four correct possibilities but does not adequately explain how they know that there are no more For example: Student finds (2, 60), (4, 30), (6, 20) [and (10, 12), perhaps implicitly] but with little or no justification (Q5).	• How could you convince me that there are no more pairs of numbers that have a greatest common factor of 2 and a least common multiple of 60?
Finds erroneous possibilities For example: Student offers (3, 40) (Q5).	• Not all of your answers are possible. How could you check them? • What is the greatest common factor of 3 & 40?
Answers all questions correctly with valid explanations Student needs an extension task.	• Can you find a set of 3 numbers that have a greatest common factor of 2 and a least common multiple of 60? And another?

SUGGESTED LESSON OUTLINE

Whole-class introduction (15 minutes)

Give each student a mini-whiteboard, pen and eraser.

The nature of the whole-class introduction will be determined by information gathered from students' responses to the assessment task *Thinking of Two Numbers*.

If, for example, most students have demonstrated a reasonable grasp of the concept of greatest common factor and least common multiple but a range of different approaches have not been evident, it may be appropriate to start the lesson with an examination of the sample student responses to promote a discussion on possible calculation methods.

If the majority of students have struggled to apply their knowledge of greatest common factor and least common multiple, spend some time eliciting students' current understanding of factors and multiples.

> *On your own, write on your mini-whiteboard what a 'factor' is.*

Allow individual think time before asking students to share what they have written on their mini-whiteboards. Once students have had some time to work individually, ask them to share with a partner.

> *Show what you have written to your neighbor. Do you agree on what a 'factor' is?*
>
> *Have you described it in a similar way? If not, what are the differences?*

Once students have had time to discuss their understanding of the term 'factor', ask if someone has a definition that they would like to share with the rest of the class. Notice whether there are differing interpretations of the term 'factor' before moving on to the meaning of 'greatest common factor'.

When students have had chance to think about the definition of a greatest common factor on their own and discuss it with their neighbor, ask a couple of students to share their definitions with the rest of the class.

> *Now, working on your own, on your mini-whiteboard, find the greatest common factor of 10 and 18. Show your method clearly on your board.*

Spend a few minutes discussing different methods that have been used, asking two or three students to demonstrate their method to the rest of the class. Use your knowledge of the methods used for the assessment task to select students to share, especially if the method used differs from their previous work. If students have all used the same method, there will be an opportunity to look at some different approaches when discussing the sample student responses in the next part of the lesson.

Continue with similar questions to gauge students' competency with the terms 'multiple' and 'least common multiple':

> *What is the difference between a 'factor' and a 'multiple'? On your whiteboards, write what a 'multiple' is.*
>
> *What do we mean by a 'least common multiple'?*
>
> *Show me on your mini-whiteboard the least common multiple of 10 and 18.*

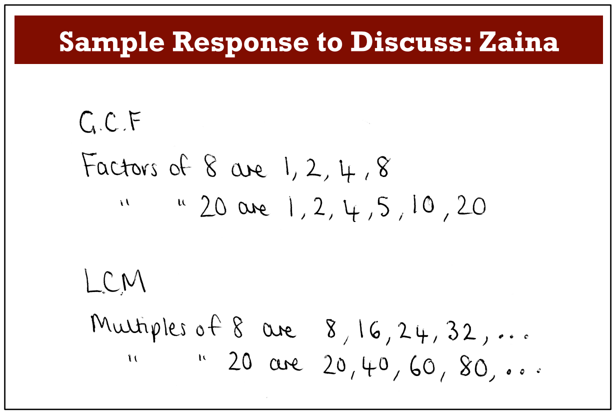
Notice whether students simply multiply the two numbers to get the least common multiple, resulting in an incorrect value of 180. Spend a few minutes exploring different methods used (and solutions if there is not a consensus).

Explain to students that they are going to be looking at some different ways of calculating the greatest common factor and least common multiple in the next part of the lesson.

Whole-class discussion: *Sample Student Responses* (15 minutes)

The purpose of the activity is for students to appreciate the strengths and weaknesses of different approaches, some of which they may or may not have encountered previously. The *Sample Student Responses* show two different methods for finding the greatest common factor and least common multiple. It is likely that students will be more familiar with Zaina's method than Aga's method. Zaina's method, however, is incomplete, as she has not actually stated the greatest common factor and least common multiple.

Display Slide P-1 of the projector resource:



If some students have used this method when completing the assessment task, acknowledge this, checking that students who have used an alternative approach are also able to follow Zaina's method:

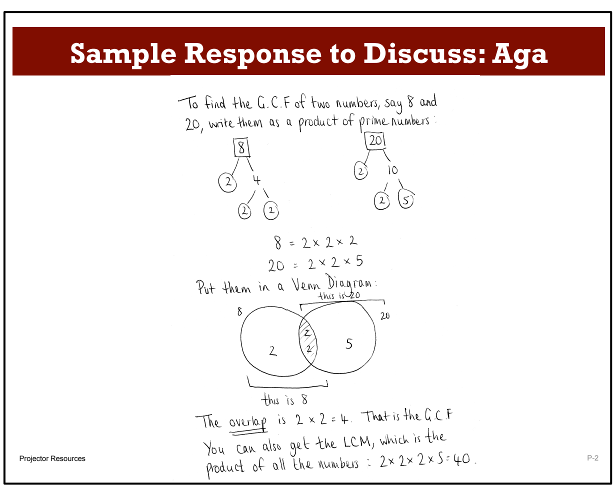
Who can explain what Zaina has done?

What does she still need to do? [Identify the GCF and LCM.]

What advice would you give Zaina about finding the least common multiple? [She needs to extend her lists of multiples until there is a multiple that is common to both lists.]

What do you like about Zaina's method?
What do you dislike?

Project Slide P-2 of projector resource:



Aga correctly finds the prime factors of 8 and 20 using factor trees and places each one in its correct position in the Venn diagram. The left circle represents 8, with the factors inside it multiplying to make 8, and the right circle represents 20. The product of the numbers in the overlap ('intersection') between the circles is the greatest common factor and the product of the numbers in the 'union' is the least common multiple. Aga obtains the greatest common factor by multiplying 2 × 2 × 2 × 5 to get 40. If students are confident with Aga's method, see if they can explain why it works:

Who can explain what Aga has done?

Why does the 'overlap' in the Venn diagram give the greatest common factor?

Can you explain why the product of all the numbers in the Venn diagram gives the least common multiple?

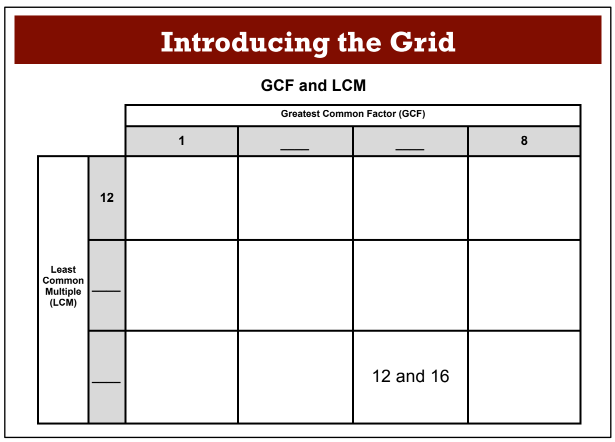
What do you like about Aga's method?
What do you dislike?

Encourage students to consider when Aga's method may not be the most useful approach. For example, could a Venn diagram be used to work out the GCF of 4 and 9? Why/ Why not?

You may decide, based on students' responses to the assessment task, that it is not appropriate to discuss both pieces of sample work. If this is the case, ensure that students have the opportunity to explore a variety of approaches as part of the introduction to the lesson.

Collaborative small-group work (20 minutes)

Give each group the grid *GCF and LCM* along with the cut-up *Card Set: Sets of Numbers* and display Slide P-3 of the projector resource:



Explain to students what they are being asked to do:

The grid is incomplete. There are some numbers missing from the shaded sections, which represent the 'greatest common factor' and 'least common multiple' for the sets of numbers that go in the empty cells. One of the cells has been completed already. This should help you to fill in some of the missing numbers.

The cards contain sets of numbers to go in the remaining empty cells. Some of the cards need completing and one of the cards is blank and you will need to fill this in yourselves. Some cells may contain more than one card.

Display Slide P-4 of the projector resource and explain to students how they are going to work together to complete the grid:

Collaborative small-group work

1. In your groups of two or three, agree on a strategy and work together to complete the grid.
2. You need to fill in the missing numbers in the shaded sections and decide which cells the sets of numbers on the cards belong to. You can do this in either order.
3. If a card is incomplete or blank, you need to complete it.
4. Each cell should contain at least one card.
5. You must all agree on the completed grid and be able to explain the position of each card in each cell of the grid.

Encourage students to use their mini-whiteboards for any calculations/working they may want to do when completing the grid.

While students are working, you have two tasks: to notice their approaches to the task and to support student reasoning.

Make a note of student approaches to the task

Notice how students make a start on the task, where they get stuck and how they respond if they do come to a halt. Do they start with the completed cell and use this to fill in missing values for the greatest common factor and least common multiple or do they start with the cards? How do they go about filling in the incomplete cards/blank card? Do they recognize that some of the cells cannot be filled nor does the 'Impossible' card surprise them? How do they respond to the card containing a set of three numbers?

Support student reasoning

Try to avoid making suggestions that push students towards a particular approach to the task. Instead, ask questions to help students clarify their thinking and explain their work:

Why does this card go in this cell? How do you know?

Why do you think it is impossible to find a set of numbers to go in this cell? How do you know?

If students are struggling to get started, you might want to suggest that they start with the completed cell. The following questions and prompts may be helpful:

What are the factors of 12? [1, 2, 3, 4, 6 and 12.]

What are the factors of 16? [1, 2, 4, 8, 16.]

Do they have any common factors? [1, 2, and 4.]

Can you tell me the greatest common factor of 12 and 16? [4.] Where will this value go in the grid?

How could you work out the least common multiple for 12 and 16?

Students who finish early could try to place other sets of three numbers into the cells. For example, (2, 3 and 4) would satisfy the top left cell, since the greatest common factor of 2, 3, and 4 is 1 and the least common multiple is 12. There are some blank cards for students to use when doing this.

Once students are happy with their completion of the grid, give them a glue stick and ask them to glue the cards onto their grid.

Whole-class discussion (10 minutes)

Hold a whole-class discussion in which students examine their approaches to completing the grid and what they have learned about greatest common factor and least common multiple by doing so.

What method did you use to complete the grid?

Which cell was the easiest to complete? Which was the most difficult? Why was this?

How did you fill in the incomplete cards?

How did you determine what numbers should go on the blank card?

Where did you place the 'Impossible' card? Why are there no numbers with this greatest common factor and least common multiple?

Where did you put the card that contained a set of three numbers?

What have you learnt about greatest common factor and least common multiple?

Follow-up lesson: reviewing the assessment task (15 minutes)

Return the original assessment task *Thinking of Two Numbers* to students, along with a copy of *Thinking of Two Numbers (revisited)*. If you have not added questions to individual pieces of work, write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

Look at your original responses and the questions (on the board/written on your paper.) Answer these questions and revise your response.

Now look at the new task sheet, Thinking of Two Numbers (revisited). Use what you have learned to answer these questions.

Some teachers give this for homework.

SOLUTIONS

Assessment task: *Thinking of Two Numbers*

1. The greatest common factor of 15 and 25 is 5.
2. The least common multiple is 75.
3. Answers will vary. Typically, students will explain that you list all the factors of both numbers and look for the greatest number that is in both lists.
4. Answers will again vary. Typically, students will explain that you list multiples of both numbers and look for the smallest number that is in both lists.
5. Leillah is wrong. Moses' two numbers **could** be 10 and 12, but they do not have to be. There are other possibilities. Students may explain that the greatest common factor of 10 and 12 is 2, because 2 is the largest number that divides into both and that the least common multiple of 10 and 12 is 60, because 60 is the smallest number that 10 and 12 go into, leaving no remainder.

   The other possibilities are (2 and 60), (4 and 30) and (6 and 20). The only pairs of integers that students need to check are those with a product of 120 (equal to the product of the GCF and LCM). Not all of these will work; e.g. 1 and 120 have a greatest common factor of 1. Alternatively, expressing 60 in terms of its prime factors as $2^2 \times 3 \times 5$ may help students to check the possibilities more efficiently, perhaps by means of a Venn diagram. Two circles can be used to represent the two numbers that we are seeking, and the 'intersection' (i.e. the greatest common factor) must be 2 while the 'union' (i.e. the least common multiple) must multiply to make 60. This method is illustrated in the sample student work by Aga.

Collaborative small-group work

		Greatest Common Factor (GCF)			
		1	2	4	**8**
Least Common Multiple (LCM)	**12**	3 and 4	4 and 6	4 and 12	Impossible
	24	3 and 8	6 and 8 2, 8 and 12	8 and 12	8 and 24
	48	16 and 3	6 and 16	**12 and 16**	16 and 24

Assessment task: *Thinking of Two Numbers (revisited)*

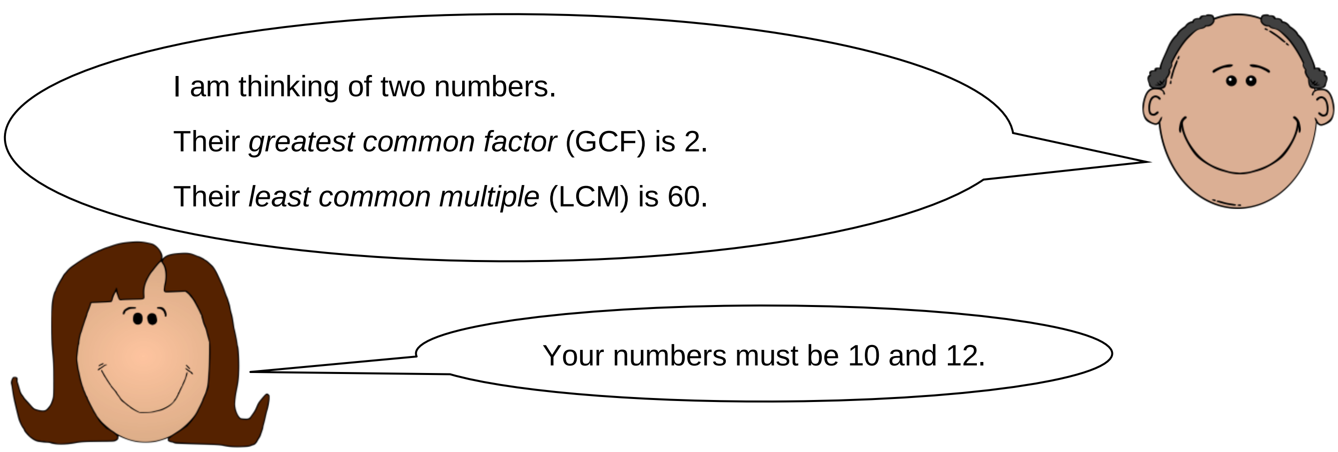
1. The greatest common factor of 16 and 24 is 8.
2. The least common multiple is 48.
3. Answers will vary. Typically, students will explain that you list all the factors of both numbers and look for the greatest number that is in both lists.
4. Answers will again vary. Typically, students will explain that you list multiples of both numbers and look for the smallest number that is in both lists.
5. Moses is wrong. Leillah's two numbers could be (9 and 30), (6 and 45), (15 and 18) and (3 and 90) so there are four possible pairs of numbers that she could be thinking of.

Thinking of Two Numbers

1. What is the *greatest common factor* of 15 and 25?
2. What is the *least common multiple* of 15 and 25?
3. In your own words, try to explain how you can find the *greatest common factor* (GCF) of any two numbers. (Use an example to explain your method, if you find it helps.)

4. In your own words, try to explain how you can find the *least common multiple* (LCM) of any two numbers. (Use an example to explain your method, if you find it helps.)

5. Moses and Leillah are playing a game called "Guess My Numbers".



Is Leillah right?
If you agree with Leillah, explain how you know that these numbers, and only these numbers, work.
If you disagree, find *all possible pairs* of numbers that Moses might be thinking of.

GCF and LCM

		Greatest Common Factor (GCF)			
		1	____	____	**8**
Least Common Multiple (LCM)	**12**				
	____				
	____			12 and 16	

Card Set: Sets of Numbers

3 and 4	3 and 8	8 and 24
8 and 12	6 and 8	6 and ___
4 and ___	16 and ___	16 and ___
Impossible	2, 8 and 12	__ and __

Card Set: Blank Cards

Thinking of Two Numbers (revisited)

1. What is the *greatest common factor* of 16 and 24?
2. What is the *least common multiple* of 16 and 24?
3. In your own words, try to explain how you can find the *greatest common factor* (GCF) of any two numbers. (Use an example to explain your method, if you find it helps.)

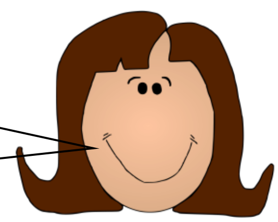
4. In your own words, try to explain how you can find the *least common multiple* (LCM) of any two numbers. (Use an example to explain your method, if you find it helps.)

5. Moses and Leillah are still playing the game "Guess My Numbers". It is Leillah's turn.

I am thinking of two numbers.

Their *greatest common factor* (GCF) is 3.

Their *least common multiple* (LCM) is 90.



There are exactly three possible pairs of numbers that you could be thinking of.

Is Moses right?
If you agree with Moses, explain how you know that there are exactly three possible pairs of numbers.
If you disagree, find all possible pairs of numbers that Leillah might be thinking of.

Sample Response to Discuss: Zaina

G.C.F

Factors of 8 are 1, 2, 4, 8

" " 20 are 1, 2, 4, 5, 10, 20

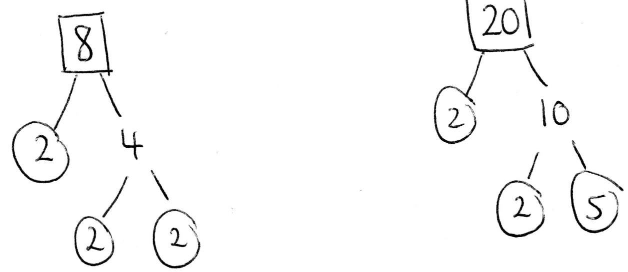
LCM

Multiples of 8 are 8, 16, 24, 32, ...

" " 20 are 20, 40, 60, 80, ...

Sample Response to Discuss: Aga

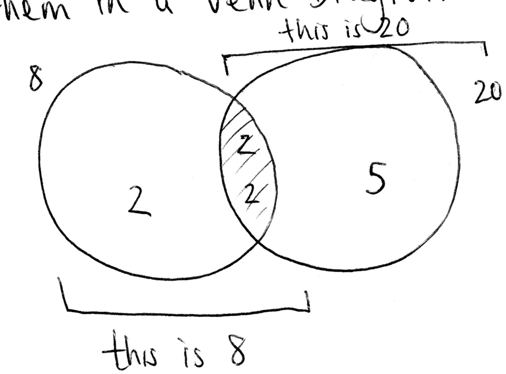
To find the G.C.F of two numbers, say 8 and 20, write them as a product of prime numbers:



$8 = 2 \times 2 \times 2$

$20 = 2 \times 2 \times 5$

Put them in a Venn Diagram:



The overlap is $2 \times 2 = 4$. That is the G.C.F

You can also get the LCM, which is the product of all the numbers: $2 \times 2 \times 2 \times 5 = 40$.

Introducing the Grid

GCF and LCM

		Greatest Common Factor (GCF)			
		1	____	____	8
Least Common Multiple (LCM)	12				
	____				
	____			12 and 16	

Collaborative small-group work

1. In your groups of two or three, agree on a strategy and work together to complete the grid.

2. You need to fill in the missing numbers in the shaded sections and decide which cells the sets of numbers on the cards belong to. You can do this in either order.

3. If a card is incomplete or blank, you need to complete it.

4. Each cell should contain at least one card.

5. You must all agree on the completed grid and be able to explain the position of each card in each cell of the grid.

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org