Concept Development Lessons

How can I help students develop a deeper understanding of Mathematics?

HANDOUTS FOR TEACHERS

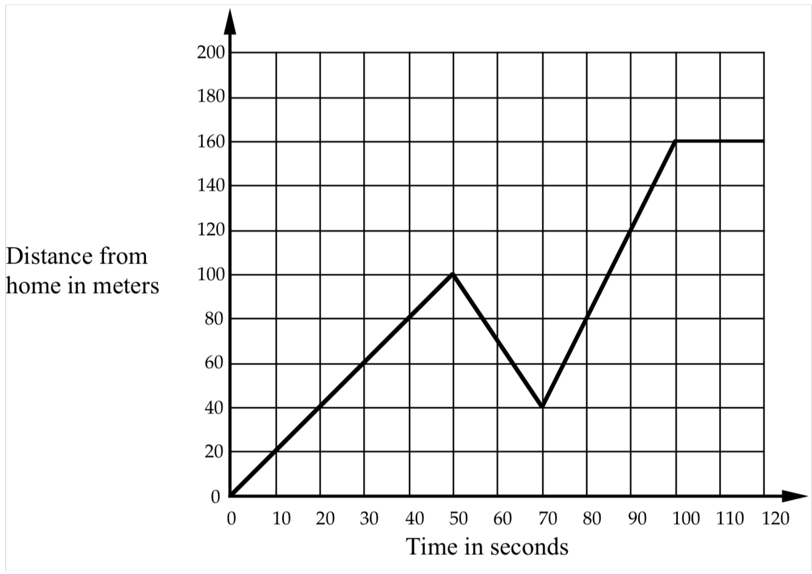
Contents

Handout 1: Assessment tasks

Assessment Task: Distance time graphs

Journey to the Bus Stop

Every morning Tom walks along a straight road from his home to a bus stop, a distance of 160 meters. The graph shows his journey on one particular day.



1. Describe what may have happened.
   You should include details like how fast he walked.

   ....

   ....

   ....

   ....

   ....

   ....

   ....

2. Are all sections of the graph realistic? Fully explain your answer.

   ....

   ....

   ....

   ....

Assessment Task: Percent changes

Percent Changes

One month Rob spent $8.02 on his phone. The next month he spent $6.00. To work out the average amount Rob spends over the two months, you could press the calculator keys:

(8 . 0 2 + 6) ÷ 2 =

1. Tom usually earns $40.85 per hour.
   He has just heard that he has had a 6% pay raise.
   He wants to work out his new pay on this calculator.
   It does not have a percent button.

   Which keys must he press on his calculator?
   Write down the keys in the correct order.
   (You do not have to do the calculation.)

2. Maria sees a dress in a sale. The dress is normally priced at $56.99.
   The ticket says that there is 45% off.
   She wants to use her calculator to work out how much the dress will cost.
   It does not have a percent button.

   Which keys must she press on her calculator?
   Write down the keys in the correct order.
   (You do not have to do the calculation.)

3. Last year, the price of an item was $350. This year it is $450.
   Lena wants to know what the percentage change is.
   Write down the calculation she will need to do to get the correct answer.
   (You do not have to do the calculation.)

4. In a sale, the prices in a shop were all decreased by 20%.
   After the sale they were all increased by 25%.
   What was the overall effect on the shop prices?
   Explain how you know.

Assessment Task: Interpreting Expressions

Interpreting Expressions

1. Write algebraic expressions for each of the following:

   a. Multiply n by 5 then add 4. ______________________

   b. Add 4 to n then multiply your answer by 5. ______________________

   c. Add 4 to n then divide your answer by 5. ______________________

   d. Multiply n by n then multiply your answer by 3. ______________________

   e. Multiply n by 3 then square your answer. ______________________

2. Imagine you are a teacher. Decide whether the following work is correct or incorrect.
   If you see an error:

   a. Cross it out and replace it with a correct answer.

   b. Explain the error using words or diagrams.

$$2(n+3) = 2n + 3$$

$$\frac{10n-5}{5} = 2n - 1$$

$$(5n)^2 = 5n^2$$

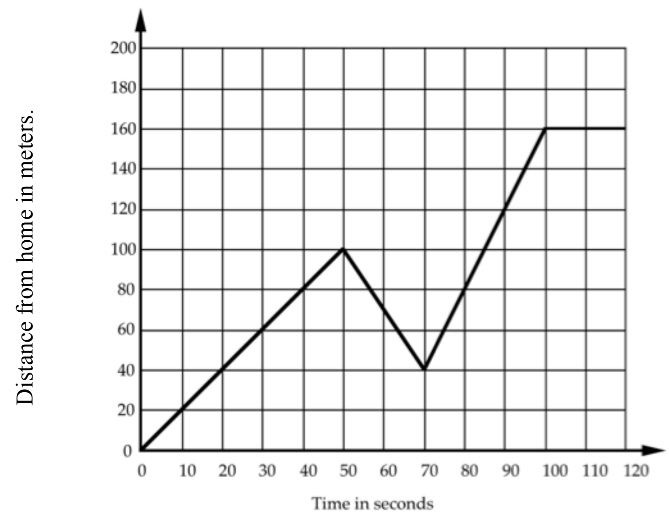
$$(n+3)^2 = n^2 + 3^2 = n^2 + 9$$

Handout 2: Sample student work

Interpreting a distance–time graph

Every morning Tom walks along a straight road from his home to a bus stop, a distance of 160 meters. The graph shows his journey on one particular day.

1. Describe what may have happened.
   You should include details like how fast he walked.



Jodie's response

Tom walked along a road for 100metres instead of walking another 30metres he took a short cut down an alleyway which took he 20 mintes he walked very quickly then he caught the bus to his college which took about 50minutes

Maxine's response

when he get out he starts walking fast to the bus stop then he slows down the he picks up the speed again and then ~~the~~ this speed goes ~~cons~~ constant.

Percent changes

1. Maria sees a dress in a sale. The dress is normally priced at $56.99.
   The ticket says that there is 45% off.
   She wants to use her calculator to work out how much the dress will cost.
   It does not have a percent button.

   Which keys must she press on her calculator?
   Write down the keys in the correct order.
   (You do not have to do the calculation.)

2. In a sale, the prices in a shop were all decreased by 20%.
   After the sale they were all increased by 25%.
   What was the overall effect on the shop prices?
   Explain how you know.

George's response

① 56·99 − 0·45

② Prices went up 5%
I know this because 25% − 20% = 5%.

Jurgen's response

1. 56·99 ÷ 100 × 45 =
~~56·99 − Ans =~~
56·99 − 56·99 ÷ 100 × 45 =

2. $56·99 = 100%
1% = 56·99 ÷ 100 = 0·5699
20% = 0·5699 × 20 = 11·398
25% = 0·5699 × 25 = 14·2475
Difference = 2·8495
$2·85

Interpreting expressions

Britney's response

Interpreting Expressions

1. Write algebraic expressions for each of the following:

   a. Multiply n by 5 then add 4. $n \times 5 + 4$

   b. Add 4 to n then multiply your answer by 5. $n + 4 \times 5$

   c. Add 4 to n then divide your answer by 5. $n + 4 \div 5$

   d. Multiply n by n then multiply your answer by 3. $n \times n \times 3$

   e. Multiply n by 3 then square your answer. $n \times 3^2$

2. Imagine you are a teacher. Decide whether the following work is correct or incorrect. If you see an error:

   a. Cross it out and replace it with a correct answer.

   b. Explain the error using words or diagrams.

$2(n+3) = 2n+3$ ✓ Correct

$\dfrac{10n-5}{5} = 2n-1$? Dont know

$(5n)^2 = 5n^2$ ✓ Correct

$(n+3)^2 = n^2 + 3^2 = n^2 + 9$? Dont know.

Handout 3: Sample follow-up questions

Distance-time graphs: Common issues

Common issues:	Suggested questions and prompts:
Student interprets the graph as a picture For example: The student assumes that as the graph goes up and down, Tom's path is going up and down. Or: The student assumes that a straight line on a graph means that the motion is along a straight path. Or: The student thinks the negative slope means Tom has taken a detour.	• *If a person walked in a circle around their home, what would the graph look like?* • *If a person walked at a steady speed up and down a hill, directly away from home, what would the graph look like?* • *In each section of his journey, is Tom's speed steady or is it changing? How do you know?* • *How can you figure out Tom's speed in each section of the journey?*
Student interprets graph as speed–time The student has interpreted a positive slope as speeding up and a negative slope as slowing down.	• *If a person walked for a mile at a steady speed, away from home, then turned round and walked back home at the same steady speed, what would the graph look like?* • *How does the distance change during the second section of Tom's journey? What does this mean?* • *How does the distance change during the last section of Tom's journey? What does this mean?* • *How can you tell if Tom is traveling away from or towards home?*
Student fails to mention distance or time For example: The student has not mentioned how far away from home Tom has traveled at the end of each section. Or: The student has not mentioned the time for each section of the journey.	• *Can you provide more information about how far Tom has traveled during different sections of his journey?* • *Can you provide more information about how much time Tom takes during different sections of his journey?*
Student fails to calculate and represent speed For example: The student has not worked out the speed of some/all sections of the journey. Or: The student has written the speed for a section as the distance covered in the time taken, such as "20 meters in 10 seconds."	• *Can you provide information about Tom's speed for all sections of his journey?* • *Can you write his speed as meters per second?*
Student misinterprets the scale For example: When working out the distance the student has incorrectly interpreted the vertical scale as going up in tens rather than twenties.	• *What is the scale on the vertical axis?*
Student adds little explanation as to why the graph is or is not realistic	• *What is the total distance Tom covers? Is this realistic for the time taken? Why?/Why not?* • *Is Tom's fastest speed realistic? Is Tom's slowest speed realistic? Why?/Why not?*

Percent changes: Common issues

Common issues:	Suggested questions and prompts:
Student makes the incorrect assumption that a percentage increase means the calculation must include an addition For example: 40.85 + 0.6 or 40.85 + 1.6. (Q1.) *A single multiplication by 1.06 is enough.*	• *Does your answer make sense? Can you check that it is correct?* • *"Compared to last year 50% more people attended the festival." What does this mean? Describe in words how you can work out how many people attended the festival this year. Give me an example.* • *Can you express the increase as a single multiplication?*
Student makes the incorrect assumption that a percentage decrease means the calculation must include a subtraction For example: 56.99 − 0.45 or 56.99 − 1.45. (Q2.) *A single multiplication by 0.55 is enough.*	• *Does your answer make sense? Can you check that it is correct?* • *In a sale, an item is marked "50% off." What does this mean? Describe in words how you calculate the price of an item in the sale. Give me an example.* • *Can you express the decrease as a single multiplication?*
Student converts the percentage to a decimal incorrectly For example: 40.85 × 0.6. (Q1.)	• *How can you write 50% as a decimal? How can you write 5% as a decimal?*
Student uses inefficient method For example: First the student calculates 1%, then multiplies by 6 to find 6%, and then adds this answer on: (40.85 ÷ 100) × 6 + 40.85. (Q1.) Or: 56.99 × 0.45 = ANS, then 56.99 − ANS (Q2.) *A single multiplication is enough.*	• *Can you think of a method that reduces the number of calculator key presses?* • *How can you show your calculation with just one step?*
Student is unable to calculate percentage change For example: 450 − 350 = 100% (Q3.) Or: The difference is calculated, then the student does not know how to proceed or he/she divides by 450. (Q3.) *The calculation (450 − 350) ÷ 350 × 100 is correct.*	• *Are you calculating the percentage change to the amount $350 or to the amount $450?* • *If the price of a t-shirt increased by $6, describe in words how you could calculate the percentage change. Give me an example. Use the same method in Q3.*
Student subtracts percentages For example: 25 − 20 = 5%. (Q4.) *Because we are combining multipliers: 0.8 × 1.25 = 1, there is no overall change in prices.*	• *Make up the price of an item and check to see if your answer is correct.*
Student fails to use brackets in the calculation For example: 450 – 350 ÷ 350 × 100. (Q4.)	• *In your problem, what operation will the calculator carry out first?*
Student misinterprets what needs to be included the answer For example: The answer is just operator symbols.	• *If you just entered these symbols into your calculator would you get the correct answer?*

Interpreting expressions: Common issues

<table>
<tr><th>Common issues:</th><th>Suggested questions and prompts:</th></tr>
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Student writes expressions left to right, showing little understanding of the order of operations implied by the symbolic representation.

For example:

Q1a Writes $n \times 5 + 4$ (not incorrect).
Q1b Writes $4 + n \times 5$.
Q1c Writes $4 + n \div 5$.
Q1d Writes $n \times n \times 3$.

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- *Can you write answers to the following?*
4 + 1 × 5
4 + 2 × 5
4 + 3 × 5
- *Check your answers with your calculator.*
How is your calculator working these out?
- *So what does 4 + n × 5 mean?*
Is this the same as Q1b?

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Student does not construct parentheses correctly or expands them incorrectly.

For example:

Q1b Writes $4 + n \times 5$ instead of $5(n + 4)$.

Q1c Writes $4 + n \div 5$ instead of $\frac{4+n}{5}$.

Q2 $2(n+3) = 2n+3$ is counted as correct.
Q2 $(5n)^2 = 5n^2$ is counted as correct.
Q2 $(n+3)^2 = n^2 + 3^2$ is counted as correct.

</td><td>

- *Which one of the following is the odd one out and why?*

  Think of a number, add 3, and then multiply your answer by 2.

  Think of a number, multiply it by 2, and then add 3.

  Think of a number, multiply it by 2, and then add 6.

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Student identifies errors but does not give explanations.

In question 2, there are corrections to the first, third, and fourth statements, but no explanation or diagram is used to explain why they are incorrect.

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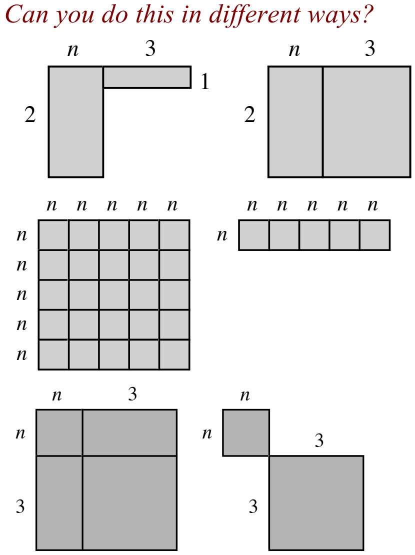
- *How would you write down expressions for these areas?*
Can you do this in different ways?



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Handout 4: Generalizations commonly made by students

What other examples can you add to this list?
Can you think of any misconceptions you have had at some time?
How were these overcome?

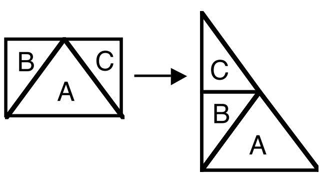
0.567 > 0.85
The more digits a number has, the larger is its value.

3 ÷ 6 = 2
You always divide the larger number by the smaller one.

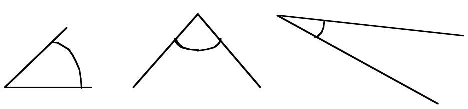
0.4>0.62
The fewer the number of digits after the decimal point, the larger is its value.
It's like fractions.

5.62 x 0.65 > 5.62
Multiplication always makes numbers bigger.

1 gallon costs $5.60; 4.2 gallons cost $5.60 x 4.2; 0.22 gallons cost $5.60 ÷ 0.22
If you change the numbers in a question, you change the operation you have to do.



Area of rectangle ≠ Area of triangle
If you dissect a shape and rearrange the pieces, you change the area.



A **B** **C**

Angle A is greatest. Angle C is greatest.
The size of an angle is related to the size of the arc or the length of the arms of the angle.

If x+4 < 10, then x = 5.
Letters represent particular numbers.

3 + 4 = 7 + 2 = 9 + 5 = 14
Equals' means 'makes'.

In three rolls of a die, it is harder to get 6,6,6 than 2,4,6.
Special outcomes are less likely than more representative outcomes.

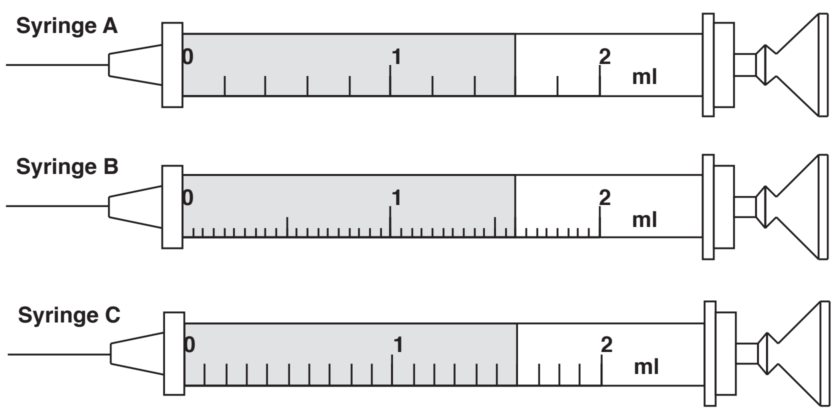
Handout 5: Principles to discuss

These principles are backed up by research evidence.
Discuss the implications for your own teaching.

- Teaching approaches that encourage the exploration of misconceptions through discussions result in deeper, longer-term learning than approaches that try to avoid mistakes by explaining the 'right way' to see things from the start.

- It is helpful if discussions focus on known difficulties. Rather than posing long lists of questions, it is better to focus on a challenging task and encourage a variety of interpretations to emerge, so that students can compare and evaluate their ideas.

- Questions can be juxtaposed in ways that create a tension (sometimes called a 'cognitive conflict') that needs resolving. Contradictions arising from conflicting methods or opinions create awareness that something needs to be learned. For example, asking students to say how much medicine is in each of the following syringes may result in answers such as "1.3ml, 1.12ml and 1.6ml". "But these quantities are all the same!" This provides a start for a useful discussion on the denary nature of decimal notation.



- Activities should provide opportunities for meaningful feedback. This does not mean providing summative information, such as the number of correct or incorrect answers. More helpful feedback is provided when students compare results obtained from alternative methods until they realize why they get different answers.

- Sessions include time for whole group discussion in which new ideas and concepts are allowed to emerge. This requires sensitivity so that students are encouraged to share tentative ideas in a non-threatening environment.

- Opportunities should be provided for students to 'consolidate' what has been learned through the application of the newly constructed concept.

Handout 6: Structure of the Concept lessons

Broadly speaking, each Concept Formative Assessment Lesson is structured in the following way, with some variation, depending on the topic and task:

- **(Before the lesson) Students complete an assessment task individually**
  This assessment task is designed to clarify students' existing understandings of the concepts under study. The teacher assesses a sample of these and plans appropriate questions that will move student thinking forward. These questions are then introduced in the lesson at appropriate points.

- **Whole class introduction**
  Each lesson begins with the teacher presenting a problem for class discussion. The aim here is to intrigue students, provoke discussion and/or model reasoning,

- **Collaborative work on a substantial activity**
  At this point the main activity is introduced. This activity is designed to be a rich, collaborative learning experience. It is both accessible and challenging; having multiple entry points and multiple solution paths. It is usually done with shared resources and is presented on a poster.
  Four types of activity are commonly used as shown in **Handout 6**. Students are involved in:

    - classifying mathematical objects & challenging definitions
    - interpreting multiple representations
    - evaluating conjectures and assertions
    - modifying situations & exploring their structure

  These will be explored more fully later in this module. It is not necessary for every student to complete the activity. Rather we hope that students will come to understand the concepts more clearly.

- **Students share their thinking with the whole class**
  Students now share some of their learning with other students. It is through explaining that students begin to clarify their own thinking. The teacher may then ask further questions to provoke deeper reflection.

- **Students revisit the assessment task**
  Finally, students are asked to look again at their original answers to the assessment task. They are either asked to improve their responses or are asked to complete a similar task. This helps both the teacher and the student to realize what has been learned from the lesson.

Handout 7: Some genres of activity used in the Concept lessons

The main activities in the concept lessons are built around the following four genres. Each of these types of activity is designed to provoke students to reason in different ways; to recognize properties, to define, to represent, to challenge conjectures and misconceptions, to recognize deeper structures in problems.

1. **Classifying mathematical objects**
   Mathematics is full of conceptual 'objects' such as numbers, shapes, and functions. In this type of activity, students examine objects carefully, and classify them according to their different attributes. Students have to select an object, discriminate between that object and other similar objects (what is the same and what is different?) and create and use categories to build definitions. This type of activity is therefore powerful in helping students understand different mathematical terms and symbols, and the process by which they are developed.

2. **Interpreting multiple representations**
   Mathematical concepts have many representations; words, diagrams, algebraic symbols, tables, graphs and so forth. These activities allow different representations to be shared, interpreted, compared and grouped in ways that allow students to construct meanings and links between the underlying concepts.

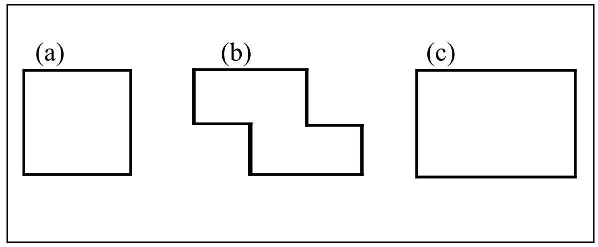
3. **Evaluating mathematical statements**
   These activities offer students a number of mathematical statements or generalizations. These statements may typically arise from student misconceptions, for example: "The square root of a number is smaller than the number." Students are asked to decide on their validity and give explanations for their decisions. Explanations usually involve generating examples and counterexamples to support or refute the statements. In addition, students may be invited to add conditions or otherwise revise the statements so that they become 'always true'.

4. **Exploring the structure of problems**
   In this type of activity, students are given the task of devising their own mathematical problems. They try to devise problems that are both challenging and that they know they can solve correctly. Students first solve their own problems and then challenge other students to solve them. During this process, they offer support and act as 'teachers' when the problem solver becomes stuck. Creating and solving problems may also be used to illustrate doing and undoing processes in mathematics. For example, one student might draw a circle and calculate its area. This student is then asked to pass the result to a neighbor, who must now try to reconstruct the circle from the given area. Both students then collaborate to see where mistakes have arisen.

Handout 8: Classifying mathematical objects

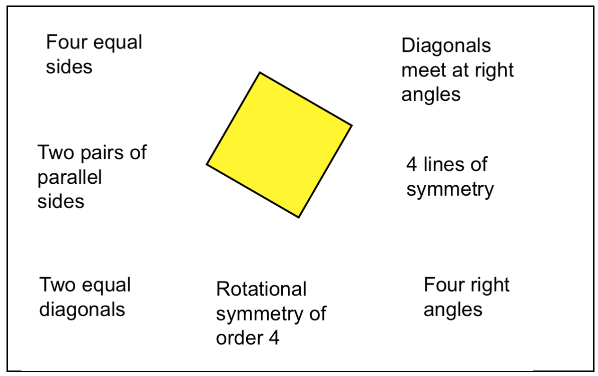
Similarities and differences

Show students three objects.
"Which is the odd one out?"
"Describe properties that two share that the third does not."
"Now choose a different object from the three and justify it as the odd one out."



Properties and definitions

Show students an object.
"Look at this object and write down all its properties."
"Does any *single* property constitute a *definition* of the object? If not, what other object has that property?"
"Which *pairs* of properties constitute a definition and which pairs do not?"



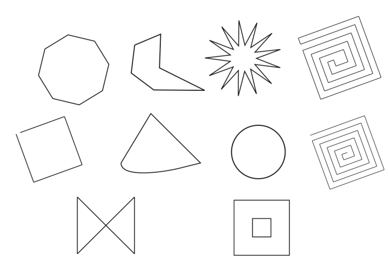
Creating and testing a definition

Ask students to write down the definition of a polygon, or some other mathematical word.

"Exchange definitions and try to improve them."

Show students a collection of objects.
"Use your definition to sort the objects."
"Now improve your definitions."

Which of these is a polygon according to your definition?



Classifying using a two-way table

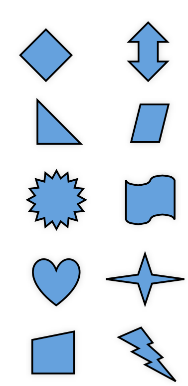
Give students a two-way table to sort a collection of objects.

"Create your own objects and add these to the table."

"Try to justify why particular entries are impossible to fill."

“Classify the objects according to your own categories.
Hide your category headings.
Can your partner identify the headings from the way you have sorted the objects?”

	No rotational symmetry	**Rotational symmetry**
No lines of symmetry		
One or two lines of symmetry		
More than two lines of symmetry		



Handout 9: Interpreting multiple representations

Each group of students is given a set of cards. They are invited to sort the cards into sets, so that each set of cards have equivalent meaning. As they do this, they have to explain how they know that cards are equivalent. They also construct for themselves any cards that are missing. The cards are designed to force students to discriminate between commonly confused representations.

Card Set A: Algebra expressions

E1 $\dfrac{n+6}{2}$	**E2** $3n^2$
E3 $2n+12$	**E4** $2n+6$
E5 $2(n+3)$	**E6** $\dfrac{n}{2}+6$
E7 $(3n)^2$	**E8** $(n+6)^2$
E9 $n^2+12n+36$	**E10** $3+\dfrac{n}{2}$
E11 n^2+6	**E12** n^2+6^2
E13	**E14**

Card Set B: Verbal descriptions

W1 Multiply n by two, then add six.	**W2** Multiply n by three, then square the answer.
W3 Add six to n then multiply by two.	**W4** Add six to n then divide by two.
W5 Add three to n then multiply by two.	**W6** Add six to n then square the answer.
W7 Multiply n by two then add twelve.	**W8** Divide n by two then add six.
W9 Square n, then add six	**W10** Square n, then multiply by nine
W11	**W12**
W13	**W14**

Card Set C: Tables

T1

n	1	2	3	4
Ans	14	16	18	20

T2

n	1	2	3	4
Ans			81	144

T3

n	1	2	3	4
Ans		10	15	22

T4

n	1	2	3	4
Ans	3		27	48

T5

n	1	2	3	4
Ans			81	100

T6

n	1	2	3	4
Ans		10	12	14

T7

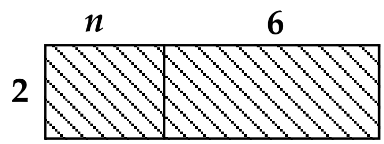
n	1	2	3	4
Ans		4		5

T8

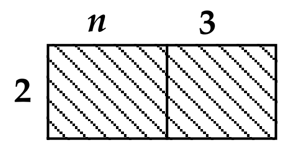
n	1	2	3	4
Ans	6.5	7	7.5	8

Card Set D: Areas

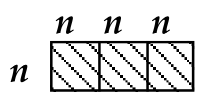
A1



A2



A3



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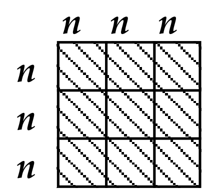
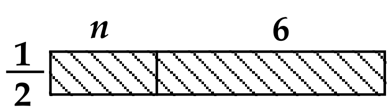


A5



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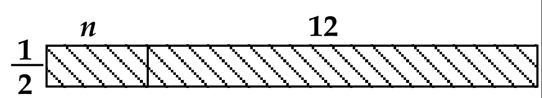

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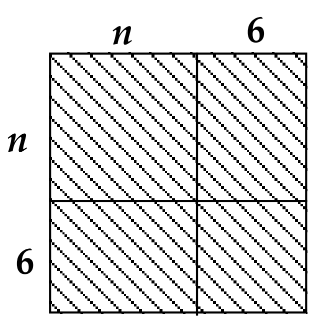

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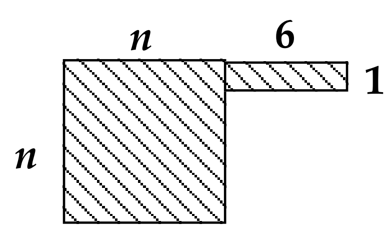

Swan, M. (2008), *A Designer Speaks: Designing a Multiple Representation Learning Experience in Secondary Algebra*. Educational Designer: Journal of the International Society for Design and Development in Education, 1(1), article 3.

Handout 10: Evaluating mathematical statements

Each group of students is given a set of statements on cards. Usually these statements are related in some way. They have to decide whether they are always, sometimes or never true.

- If they think it is *always* or *never* true, then they must try to explain how they can be sure.
- If they think it is *sometimes* true, they must define exactly when it is true and when it is not.

Pay rise Max gets a pay rise of 30%. Jim gets a pay rise of 25%. So Max gets the bigger pay rise.	**Sale** In a sale, every price was reduced by 25%. After the sale every price was increased by 25%. So prices went back to where they started.
Area and perimeter When you cut a piece off a shape you reduce its area and perimeter.	**Right angles** A pentagon has fewer right angles than a rectangle.
Birthdays In a class of ten students, the probability of two students being born on the same day of the week is one.	**Lottery** In a lottery, the six numbers 3, 12, 26, 37, 44, 45 are more likely to come up than the six numbers 1, 2, 3, 4, 5, 6.
Bigger fractions If you add the same number to the top and bottom of a fraction, the fraction gets bigger in value.	**Smaller fractions** If you divide the top and bottom of a fraction by the same number, the fraction gets smaller in value.
Square roots The square root of a number is less than or equal to the number	**Series** If the limit of the sequence of terms in an infinite series is zero, then the sum of the series is zero.

Handout 11: Students modifying a given problem

Here is a typical word problem from a textbook.

The candles problem

A student wants to earn some money by making and selling candles.
Suppose that she can make 60 candles from a $50 kit, and that these will each be sold for $4.
How much profit will she make?

After answering such a question, we might explore its structure and attempt some generalizations.
First remove all the numbers from the problem:

		k	
The cost of buying the kit: (This includes the molds, wax and wicks.)	$	50	
		n	
The number of candles that can be made with the kit:		60	candles
		s	
The price at which he sells each candle:	$	4	per candle
		p	
Total profit made if all the candles are sold:	$	190	

Now we can ask the following, first using numerical values, then using variables:

1. How did we calculate the profit *p* using the given values of *k*, *n*, and *s*?
   Would your method change if the values of *k*, *n*, and *s* were different?

2. Write in the profit and erase one of the other values: the selling price of each candle, *s*.
   How can you figure out the value of s from the remaining values of *k, n* and *p*?
   Repeat, but now erase the value of a different variable and say how it may be reconstructed from the remaining values.

3. Suppose you didn't know either of the values of *n* and *p*, but you knew the remaining values.
   How will the profit depend on the number of candles made? Plot a graph.
   Repeat for other pairs of variables.

4. Write down four general formulas showing the relationships between the variables.

   p = ……………..… *s* = ……………..… *n* = ……………..… *k* = ……………..…

Handout 12: Students creating problems for each other

Ask students to work in pairs. Each creates a problem for the other to solve.

Doing: The problem poser…	Undoing: The problem solver…
• generates an equation step-by-step, starting with, say, $x = 4$ and 'doing the same to both sides'	• solves the resulting equation: $\frac{10x+9}{8} - 7 = -0.875$
• draws a rectangle and calculates its area and perimeter.	• tries to draw a rectangle with the given area and perimeter.
• writes down an equation of the form y=mx+c and plots a graph.	• tries to find an equation that fits the resulting graph.
• expands an algebraic expression such as $(x+3)(x-2)$	• factorizes the resulting expression: • $x^2 + x - 6$
• writes down a polynomial and differentiates it	• integrates the resulting function
• writes down five numbers and finds their mean, median, range	• tries to find five numbers with the given mean, median and range.